Institute of
Applied Mechanics
Institut für Baumechanik

Graz University of Technology

Preprint Series

Institute of Applied Mechanics

Graz University of Technology

Preprint No 1/2008

Convolution Quadrature Method based symmetric Galerkin Boundary Element Method for 3-d elastodynamics

Lars Kielhorn
&
Martin Schanz

Institute of Applied Mechanics, Graz University of Technology

Abstract

The boundary integral equations in 3-d elastodynamics contain convolution integrals with respect to the time. They can be performed analytically or with the convolution quadrature method. The latter time stepping procedure's benefit is the usage of the Laplace transformed fundamental solution. Therefore, it is possible to apply this method also to problems where analytical time-dependent fundamental solutions might not be known.

To obtain a symmetric formulation the second boundary integral equation has to be used which, unfortunately, requires special care in the numerical implementation since it involves hypersingular kernel functions. Therefore, a regularization for closed surfaces of the Laplace transformed elastodynamic kernel functions is presented which transforms the bilinear form of the hypersingular integral operator to a weakly singular one. Supplementary, a weakly singular formulation of the Laplace transformed elastodynamic double layer potential is presented. This results in a time domain boundary element formulation involving at least only weakly singular integral kernels.

Finally, numerical studies validate this approach with respect to different spatial and time discretizations. Further, a comparison to the wider used collocation method is presented. It is shown numerically that the presented formulation exhibits a good convergence rate and a more stable behavior compared to collocation methods.

1 INTRODUCTION

The Boundary Element Method (BEM) in time domain is known to be well suited to treat wave propagation problems (for an overview see [3, 4]). A general mathematical background of the underlying boundary integral equations for time dependent problems may be found in [8]. Additionally, the more particular elastodynamic case is treated in [7]. Nowadays in engineering, the spatial discretization of those time-dependent boundary integral equations is mostly done via the collocation method (see, e.g. [16]). But also Galerkin type approaches exist which are mostly used for time independent problems (for an overview see [6]).

Here, the symmetric Galerkin boundary element method (SGBEM) will be applied to solve mixed initial boundary value problems of 3-d elastodynamics. Unfortunately, the drawback when considering mixed problems by symmetric methods is the evaluation of the second boundary integral equation which, in fact, contains a non integrable hypersingularity. This kind of singularity has to be interpreted as a finite part integral in the sense of Hadamard [23]. Those singularities can be either treated numerically [38] or in an analytical way [18]. Here, an analytical transformation of the hypersingular integral operator based on integration by parts in 3-d elastodynamics will be presented which, finally, will lead to a bilinear form containing only weak singularities. The used approach is very similar to a regularization in elastostatics given by Han [24] since it takes advantage of the similar structure between the elastostatic and the elastodynamic fundamental solutions.

When dealing with singularities in dynamics it is a common practice to subtract and to add the corresponding static fundamental solution from its elastodynamic counterpart. This is due to the fact that the singular behavior of the static part coincides with the dynamic one (see, e.g., [5], [25]). Hence, an existing regularization for the static case can be used to regularize also the equivalent dynamic integral kernel. Unfortunately, the occurring residual kernel might cause numerical instabilities due to the fact that it involves the difference between the singular dynamic and the singular static kernel. The method presented here does not cause those instabilities since the elastodynamic hypersingular integral operator is treated as a whole.

Regularization approaches of non integrable kernel functions based on integration by parts have a long tradition and are well known nowadays. This technique was firstly used in 1949 by Maue [33] who applied it to the wave equation in frequency domain. A major enhancement was then given by Nedelec [34] who introduced regularized hypersingular bilinear forms for the Laplace equation, the Helmholtz equation as well as for the system of linear elastostatics. Further, regularizations in the field of 3-d time-harmonic elastodynamics were presented by Nishimura & Kobayashi [36] and Becache et al. [2]. While these both approaches rely mainly on the previous work by Nedelec [34], the particular regularizations are nevertheless slightly different. In the first case the hypersingular operator is used within a collocation scheme while in the other case a Galerkin scheme is formulated. Using the latter discretization scheme is advantageous since it features less restrictions concerning the choice of shape functions for the displacement field. Contrary to the collocation approach, the only requirement in Galerkin based regularizations is the continuity of the displacement field. Another regularization of the hypersingular bilinear form in case of 3-d elastostatics was presented by Han [24] who used some basic results from Kupradze [26]. Unlike Nedelec whose regularization is based on a very general approach and, therefore, results in rather complicated formulae, Han restricts his

regularization a priori to the isotropic case and discards the possibility of describing also the anisotropic system. Hence, the resulting regularized bilinear form is simpler to deal with and motivates the use of Han's proof within the present work. As will be shown the extension of his proof to the system of 3-d elastodynamics leads to a more convenient formulation with respect to the numerical implementation than the already established regularizations [36, 2].

For the time discretization there exist in principle two approaches. Firstly, if time dependent fundamental solutions are available, the usage of ansatz functions with respect to time yields a time stepping procedure after an analytical time integration within each time step. This technique has been proposed by Mansur [32] and is denoted in the following as the *classical* time domain boundary element formulation. Secondly, the Convolution Quadrature Method (CQM) developed by Lubich [28], [29] can be used to establish the same time stepping procedure as obtained by a direct time integration [39]. Contrary to the approach from Mansur for this methodology only the Laplace domain fundamental solutions have to be used and the time integration is performed numerically. Hence, this approach can easily be extended to the viscoelastic case [40] where the fundamental solutions in closed form are only available in Laplace or Fourier domain. Moreover, the regularization process is more advantageous since the fundamental solutions in Laplace domain are simpler to deal with. This is due to the fact that no retarded potentials occur like in time-dependent fundamental solutions [20].

Here, the CQM based approach will be used in conjunction with a symmetric Galerkin boundary element formulation. This is motivated by the positive results of the Galerkin approach in elastostatics [41] as well as for parabolic problems [31] and the Helmholtz equation [30].

Finally, the present formulation will be validated and it will be studied whether the stability of the time stepping procedure is improved compared to the collocation based approach.

2 BOUNDARY INTEGRAL EQUATIONS

2.1 Formulation of the problem

Let $\Omega \subset \mathbb{R}^3$ be a domain with boundary $\Gamma = \partial\Omega$ and let the final time $T \in \mathbb{R}_+$ be fixed. A displacement of a point $\tilde{\mathbf{x}} = [\tilde{x}_1, \tilde{x}_2, \tilde{x}_3]^\mathsf{T} \in \Omega$ at the time t lying in the open interval $(0, T)$ is denoted by $\mathbf{u}(\tilde{\mathbf{x}}; t) = [u_1(\tilde{\mathbf{x}}; t), u_2(\tilde{\mathbf{x}}; t), u_3(\tilde{\mathbf{x}}; t)]^\mathsf{T}$. The displacement field $\mathbf{u}(\tilde{\mathbf{x}}; t)$ satisfies the equation of motion

$$(\mathcal{L}(\partial_{\tilde{\mathbf{x}}}) + \rho\frac{\partial^2}{\partial t^2})\mathbf{u} = \mathbf{b} \,, \tag{1}$$

where ρ is the mass density, $\mathcal{L}(\partial_{\tilde{\mathbf{x}}})$ is the Lamé-Navier operator

$$\mathcal{L}(\partial_{\tilde{\mathbf{x}}}) = -(\lambda + \mu)\nabla_{\tilde{\mathbf{x}}}\nabla_{\tilde{\mathbf{x}}} \cdot - \mu\Delta \tag{2}$$

and $\mathbf{b} = \mathbf{b}(\tilde{\mathbf{x}}; t)$ is a given force per unit volume. This body force is assumed to be absent in the following, i.e., $\mathbf{b} \equiv \mathbf{0}$ holds. In (2), λ and μ are the Lamé constants and the operator $\nabla_{\tilde{\mathbf{x}}}$ denotes the Nabla operator whereas the subscript indicates that the partial derivatives are taken with respect to the point $\tilde{\mathbf{x}}$. Moreover, Δ represents the Laplacian.

To consider a mixed initial boundary value problem, first, the boundary Γ is split into two non-overlapping sets $\Gamma_{D,i}$ and $\Gamma_{N,i}$ with respect to the i-th direction such that $\Gamma = \Gamma_{D,i} \cup \Gamma_{N,i}$

holds. By taking into account some component-wise prescribed Dirichlet data $g_{D,i}(\mathbf{x};t)$ on $\Gamma_{D,i}$ and Neumann data $g_{N,i}(\mathbf{x};t)$ on $\Gamma_{N,i}$, respectively, the system reads as

$$
\begin{aligned}
((\mathcal{L}(\partial_{\tilde{\mathbf{x}}}) + \rho\frac{\partial^2}{\partial t^2})\mathbf{u})(\tilde{\mathbf{x}};t) &= \mathbf{0} & \forall\,(\tilde{\mathbf{x}};t) &\in \Omega \times (0,T) \\
u_i(\mathbf{x};t) &= g_{D,i}(\mathbf{x};t) & \forall\,(\mathbf{x};t) &\in \Gamma_{D,i} \times (0,T) \\
t_i(\mathbf{x};t) &= g_{N,i}(\mathbf{x};t) & \forall\,(\mathbf{x};t) &\in \Gamma_{N,i} \times (0,T)\,.
\end{aligned}
\tag{3}
$$

In (3), $t_i(\mathbf{x};t)$ represents the i-th component of the traction vector $\mathbf{t}(\mathbf{x};t) = \boldsymbol{\sigma}(\mathbf{x};t) \cdot \mathbf{n}(\mathbf{x})$ defined by the product of the Cauchy stress tensor $\boldsymbol{\sigma}(\mathbf{x};t)$ with the outward unit normal vector $\mathbf{n}(\mathbf{x})$.

Moreover, homogeneous initial conditions are assumed such that $\mathbf{u}(\tilde{\mathbf{x}};0^+) = \mathbf{0}$ and $\frac{\partial}{\partial t}\mathbf{u}(\tilde{\mathbf{x}};0^+) = \mathbf{0}$ hold for all $\tilde{\mathbf{x}} \in \Omega$.

2.2 Space-time representation formulas and boundary integral equations

To obtain a boundary element formulation of the stated problem, first, an appropriate boundary integral representation of the given system of partial differential equations has to be derived. Therefore, the representation formula corresponding to (3) is introduced [13]

$$
\begin{aligned}
\mathbf{u}(\tilde{\mathbf{x}};t) = &\int_0^t \int_\Gamma \mathbf{U}(\mathbf{y}-\tilde{\mathbf{x}};t-\tau) \cdot (\mathcal{T}(\partial_{\mathbf{y}},\mathbf{n}(\mathbf{y}))\mathbf{u})(\mathbf{y};\tau)\,\mathrm{d}s_{\mathbf{y}}\,\mathrm{d}\tau \\
&- \int_0^t \int_\Gamma [(\mathcal{T}(\partial_{\mathbf{y}},\mathbf{n}(\mathbf{y}))\mathbf{U})(\mathbf{y}-\tilde{\mathbf{x}};t-\tau)]^{\mathsf{T}} \cdot \mathbf{u}(\mathbf{y};\tau)\,\mathrm{d}s_{\mathbf{y}}\,\mathrm{d}\tau \quad \forall\,\tilde{\mathbf{x}} \in \Omega, \mathbf{y} \in \Gamma, t \in (0,T)
\end{aligned}
\tag{4}
$$

containing the function $\mathbf{U}(\mathbf{y}-\tilde{\mathbf{x}};t-\tau)$ which satisfies

$$
\int_0^T \int_\Omega ((\mathcal{L}(\partial_{\mathbf{y}}) + \rho\frac{\partial^2}{\partial t^2})\mathbf{U})(\mathbf{y}-\tilde{\mathbf{x}};t-\tau) \cdot \boldsymbol{\phi}(\mathbf{y};\tau)\,\mathrm{d}\mathbf{y}\,\mathrm{d}\tau = \boldsymbol{\phi}(\tilde{\mathbf{x}};t) \quad \forall\,\tilde{\mathbf{x}},\mathbf{y} \in \Omega,\, t,\tau \in (0,T)
\tag{5}
$$

for any function ϕ. A function $\mathbf{U}(\mathbf{y}-\tilde{\mathbf{x}};t-\tau)$ with the property (5) is called fundamental solution. In fact, it is the main ingredient of any boundary element formulation.

As it can be seen, (4) contains only spatial integrals over the boundary Γ. Moreover, causality implies that all integrations with respect to the time variable must be of Volterra type. The occurring operator $\mathcal{T}(\partial_{\mathbf{y}},\mathbf{n}(\mathbf{y}))$ is a trace operator related to the outward normal vector $\mathbf{n}(\mathbf{y})$. In elastodynamics it represents the stress-strain relation based on Hooke's law. Hence, applying $\mathcal{T}(\partial_{\mathbf{y}},\mathbf{n}(\mathbf{y}))$ to the displacement field $\mathbf{u}(\mathbf{y};t)$ yields the relation

$$
(\mathcal{T}(\partial_{\mathbf{y}},\mathbf{n}(\mathbf{y}))\mathbf{u})(\mathbf{y};t) = \mathbf{t}(\mathbf{y};t) = \boldsymbol{\sigma}(\mathbf{y};t) \cdot \mathbf{n}(\mathbf{y})\,.
\tag{6}
$$

Until now, (4) holds only for a point $\tilde{\mathbf{x}} \in \Omega$. Therefore, a limiting process $\Omega \ni \tilde{\mathbf{x}} \to \mathbf{x} \in \Gamma$ has to

be performed which finally gives the first boundary integral equation

$$
\mathbf{u}(\mathbf{x};t) = \frac{1}{2}\mathbf{u}(\mathbf{x};t) - \int\limits_0^t \int\limits_\Gamma \left[(\mathcal{T}(\partial_\mathbf{y}, \mathbf{n}(\mathbf{y}))\mathbf{U})(\mathbf{y}-\mathbf{x};t-\tau) \right]^\mathsf{T} \cdot \mathbf{u}(\mathbf{y};\tau)\, \mathrm{d}s_\mathbf{y}\, \mathrm{d}\tau
$$
$$
+ \int\limits_0^t \int\limits_\Gamma \mathbf{U}(\mathbf{y}-\mathbf{x};t-\tau) \cdot \mathbf{t}(\mathbf{y};\tau)\, \mathrm{d}s_\mathbf{y}\, \mathrm{d}\tau
\tag{7}
$$

valid for almost all $\mathbf{x}, \mathbf{y} \in \Gamma$, all times $t \in (0, T)$, and a sufficiently smooth boundary Γ. In (7), the singular behavior of the kernel functions has to be considered. Hence, the first integral in (7) is weakly while the second one is strongly singular and has to be interpreted in the sense of a Cauchy principal value.

The second, or traction boundary integral equation can be derived by the application of the operator $\lim\limits_{\Omega \ni \tilde{\mathbf{x}} \to \mathbf{x} \in \Gamma} \mathcal{T}(\partial_{\tilde{\mathbf{x}}}, \mathbf{n}(\mathbf{x}))$ to (4)

$$
\mathbf{t}(\mathbf{x};t) = \frac{1}{2}\mathbf{t}(\mathbf{x};t) + \int\limits_0^t \int\limits_\Gamma (\mathcal{T}(\partial_\mathbf{x}, \mathbf{n}(\mathbf{x}))\mathbf{U})(\mathbf{y}-\mathbf{x};t-\tau) \cdot \mathbf{t}(\mathbf{y};\tau)\, \mathrm{d}s_\mathbf{y}\, \mathrm{d}\tau
$$
$$
- \int\limits_0^t \lim\limits_{\Omega \ni \tilde{\mathbf{x}} \to \mathbf{x} \in \Gamma} \mathcal{T}(\partial_{\tilde{\mathbf{x}}}, \mathbf{n}(\mathbf{x})) \int\limits_\Gamma \left[(\mathcal{T}(\partial_\mathbf{y}, \mathbf{n}(\mathbf{y}))\mathbf{U})(\mathbf{y}-\tilde{\mathbf{x}};t-\tau) \right]^\mathsf{T} \cdot \mathbf{u}(\mathbf{y};\tau)\, \mathrm{d}s_\mathbf{y}\, \mathrm{d}\tau \, .
\tag{8}
$$

Note, that the limiting process to obtain (8) is the same as for the first boundary integral equation (7). Due to the singular kernels in (7) the first integral in (8) is strongly singular while the second one contains a hypersingularity which has to be understand in the sense of a finite part. This hypersingular kernel will be discussed in detail in section 4.

Now, by introducing the operators

$$
(I * \mathbf{w})_\Gamma(\mathbf{x};t) = \int\limits_0^t \int\limits_\Gamma \delta(\mathbf{y}-\mathbf{x};t-\tau)\mathbf{I} \cdot \mathbf{w}(\mathbf{y};\tau)\, \mathrm{d}s_\mathbf{y}\, \mathrm{d}\tau
$$

$$
(\mathcal{V} * \mathbf{w})_\Gamma(\mathbf{x};t) = \int\limits_0^t \int\limits_\Gamma \mathbf{U}(\mathbf{y}-\mathbf{x};t-\tau) \cdot \mathbf{w}(\mathbf{y};\tau)\, \mathrm{d}s_\mathbf{y}\, \mathrm{d}\tau
$$

$$
(\mathcal{K}' * \mathbf{w})_\Gamma(\mathbf{x};t) = \int\limits_0^t \int\limits_\Gamma (\mathcal{T}(\partial_\mathbf{x}, \mathbf{n}(\mathbf{x}))\mathbf{U})(\mathbf{y}-\mathbf{x};t-\tau) \cdot \mathbf{w}(\mathbf{y};\tau)\, \mathrm{d}s_\mathbf{y}\, \mathrm{d}\tau
$$

$$
(\mathcal{K} * \mathbf{w})_\Gamma(\mathbf{x};t) = \int\limits_0^t \int\limits_\Gamma \left[(\mathcal{T}(\partial_\mathbf{y}, \mathbf{n}(\mathbf{y}))\mathbf{U})(\mathbf{y}-\mathbf{x};t-\tau) \right]^\mathsf{T} \cdot \mathbf{w}(\mathbf{y};\tau)\, \mathrm{d}s_\mathbf{y}\, \mathrm{d}\tau
$$

$$
(\mathcal{D} * \mathbf{w})_\Gamma(\mathbf{x};t) = - \int\limits_0^t \lim\limits_{\Omega \ni \tilde{\mathbf{x}} \to \mathbf{x} \in \Gamma} \mathcal{T}(\partial_{\tilde{\mathbf{x}}}, \mathbf{n}(\mathbf{x})) \int\limits_\Gamma \left[(\mathcal{T}(\partial_\mathbf{y}, \mathbf{n}(\mathbf{y}))\mathbf{U})(\mathbf{y}-\tilde{\mathbf{x}};t-\tau) \right]^\mathsf{T} \cdot \mathbf{w}(\mathbf{y};\tau)\, \mathrm{d}s_\mathbf{y}\, \mathrm{d}\tau
\tag{9}
$$

for some arbitrary functions $\mathbf{w}(\mathbf{y}; \tau)$ the equations (7) and (8) can be written more compact

$$\left[\begin{pmatrix} \frac{1}{2}I - \mathcal{K} & \mathcal{V} \\ \mathcal{D} & \frac{1}{2}I + \mathcal{K}' \end{pmatrix} * \begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix} \right]_\Gamma (\mathbf{x}; t) = \begin{pmatrix} \mathbf{u}(\mathbf{x}; t) \\ \mathbf{t}(\mathbf{x}; t) \end{pmatrix} . \tag{10}$$

In (9) and (10), the $*$ abbreviates the convolution integrals in time. Beside the identity given in the first line of (9) the operators can sequentially be titled as single layer, adjoint double layer, and double layer potential. The last operator is the so-called hypersingular integral operator.

2.3 Symmetric Galerkin formulation

To find the complete Cauchy data $[\mathbf{u}, \mathbf{t}]^\mathsf{T}$ the symmetric formulation as proposed in [41, 9] is used. Therefore, the first boundary integral equation (7) is used only on the Dirichlet part $\Gamma_{D,i}$ while the second one (8) is used on the Neumann part $\Gamma_{N,i}$. Note, because here a vectorial problem has to be solved at each boundary point different types of boundary data in each direction $i = 1, 2, 3$ may be prescribed. This yields

$$(\mathcal{V} * \mathbf{t})_{\Gamma_i}(\mathbf{x}; t) - (\mathcal{K} * \mathbf{u})_{\Gamma_i}(\mathbf{x}; t) = \frac{1}{2} g_{D,i}(\mathbf{x}; t) \quad \forall \, \mathbf{x} \in \Gamma_{D,i}$$
$$(\mathcal{K}' * \mathbf{t})_{\Gamma_i}(\mathbf{x}; t) + (\mathcal{D} * \mathbf{u})_{\Gamma_i}(\mathbf{x}; t) = \frac{1}{2} g_{N,i}(\mathbf{x}; t) \quad \forall \, \mathbf{x} \in \Gamma_{N,i} . \tag{11}$$

Now, the displacements $u_i(\mathbf{y}; t)$ as well as the tractions $t_i(\mathbf{y}; t)$ are decomposed into

$$\begin{aligned} u_i &= \tilde{u}_i + \tilde{g}_{D,i} & \text{with} \quad \tilde{u}_i = 0, \; \tilde{g}_{D,i} = g_{D,i} & \forall \, \mathbf{y} \in \Gamma_{D,i} \\ t_i &= \tilde{t}_i + \tilde{g}_{N,i} & \text{with} \quad \tilde{t}_i = 0, \; \tilde{g}_{N,i} = g_{N,i} & \forall \, \mathbf{y} \in \Gamma_{N,i} \end{aligned} \tag{12}$$

where $\tilde{g}_{D,i}$ and $\tilde{g}_{N,i}$ are arbitrary but fixed extensions of the prescribed boundary data. Note, that the extensions $\tilde{u}_i$ and $\tilde{g}_{D,i}$ have to be continuous due to regularity requirements [43]. After inserting the decompositions (12) into the system of boundary integral equations (11) the symmetric boundary integral formulation

$$(\mathcal{V} * \tilde{\mathbf{t}})_{\Gamma_{D,i}}(\mathbf{x}; t) - (\mathcal{K} * \tilde{\mathbf{u}})_{\Gamma_{N,i}}(\mathbf{x}; t) = ((\tfrac{1}{2}I + \mathcal{K}) * \tilde{\mathbf{g}}_D)_{\Gamma_i}(\mathbf{x}; t) - (\mathcal{V} * \tilde{\mathbf{g}}_N)_{\Gamma_i}(\mathbf{x}; t)$$
$$(\mathcal{K}' * \tilde{\mathbf{t}})_{\Gamma_{D,i}}(\mathbf{x}; t) + (\mathcal{D} * \tilde{\mathbf{u}})_{\Gamma_{N,i}}(\mathbf{x}; t) = ((\tfrac{1}{2}I - \mathcal{K}') * \tilde{\mathbf{g}}_N)_{\Gamma_i}(\mathbf{x}; t) - (\mathcal{D} * \tilde{\mathbf{g}}_D)_{\Gamma_i}(\mathbf{x}; t) \tag{13}$$

is obtained with the unknown Cauchy data $[\tilde{\mathbf{u}}, \tilde{\mathbf{t}}]^\mathsf{T}$. By using the inner product $\langle f, g \rangle_\Gamma = \int_\Gamma f(\mathbf{x}) g(\mathbf{x}) \, ds_\mathbf{x}$ a variational formulation is introduced to find $[\tilde{\mathbf{u}}, \tilde{\mathbf{t}}]^\mathsf{T}$ such that

$$\langle \mathcal{V} * \tilde{\mathbf{t}}, \mathbf{w} \rangle_{\Gamma_{D,i}} - \langle \mathcal{K} * \tilde{\mathbf{u}}, \mathbf{w} \rangle_{\Gamma_{D,i}} = \langle (\tfrac{1}{2}I + \mathcal{K}) * \tilde{\mathbf{g}}_D - \mathcal{V} * \tilde{\mathbf{g}}_N, \mathbf{w} \rangle_{\Gamma_{D,i}}$$
$$\langle \mathcal{K}' * \tilde{\mathbf{t}}, \mathbf{v} \rangle_{\Gamma_{N,i}} + \langle \mathcal{D} * \tilde{\mathbf{u}}, \mathbf{v} \rangle_{\Gamma_{N,i}} = \langle (\tfrac{1}{2}I - \mathcal{K}') * \tilde{\mathbf{g}}_N - \mathcal{D} * \tilde{\mathbf{g}}_D, \mathbf{v} \rangle_{\Gamma_{N,i}} \tag{14}$$

holds for all test-functions $\mathbf{w}(\mathbf{x}), \mathbf{v}(\mathbf{x})$. Note, as the Galerkin scheme is used only for the spatial integrations the test-functions $\mathbf{w}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ exhibit no time dependency.

3 BOUNDARY ELEMENT FORMULATION

3.1 Time discretization

The CQM as proposed by Lubich [28, 29] is used to approximate the convolution integrals in time. As already stated in section 1, this approach is advantageous due to the fact that the CQM deals only with the Laplace transformed fundamental solutions, which makes it also attractive for handling problems where fundamental solutions are only known in Laplace domain or are not given in closed form in the time domain.

   The goal is the computation of the convolution integral

$$y(t) = f * g = \int_0^t f(t-\tau)\, g(\tau)\, \mathrm{d}\tau\,. \tag{15}$$

If the Laplace transform of the function $f(t)$ is known and by dividing the time into M intervals of equal step size Δt the integral can be approximated for the time step $t_m = m\Delta t$ by

$$y_m = y(m\Delta t) \approx \sum_{k=0}^m \omega_{m-k}(\hat{f}, \Delta t)\, g(k\Delta t) \tag{16}$$

with the integration weights ω_n. These integration weights are given by

$$\omega_n(\hat{f}, \Delta t) = \frac{R^{-n}}{2\pi} \int_0^{2\pi} \hat{f}\left(\frac{\gamma\left(Re^{i\varphi}\right)}{\Delta t}\right) e^{-in\varphi}\, \mathrm{d}\varphi \tag{17}$$

and can be approximated by the trapezoidal rule

$$\omega_n(\hat{f}, \Delta t) = \frac{1}{L} \sum_{\ell=0}^{L-1} \hat{f}\left(\frac{\gamma(\zeta_\ell)}{\Delta t}\right) \zeta_\ell^{-n}\,, \quad \text{with} \quad \zeta_\ell := Re^{i\ell\frac{2\pi}{L}}. \tag{18}$$

From (18), it is obvious that the integration weights depend only on the Laplace transform of the function f denoted by $\hat{f}(s) = \mathscr{L}\{f\}(s)$ with the complex Laplace variable $s \in \mathbb{C}$. The parameters R and L depend mainly on the number of time steps M and are chosen as $R = 10^{-5/2M}$ and $L = M$. Moreover, this choice allows the computation of the integration weights via a technique similar to the Fast Fourier Transformation. Finally, the term $\gamma(\cdot)$ represents the characteristic function of an underlying multistep method, e.g., a backward differential formula of order 2 (BDF2) with $\gamma(s) = 3/2 - 2s + s^2/2$. More details about the Convolution Quadrature Method and the choice of the used parameters can be found in [28], [29], and [39].

   Now, the convolution quadrature formula (16) with the definition of the integration weights given in (18) is applied to the variational formulation (14). For example, an entry corresponding to the single layer potential can be computed for t_m as

$$
\langle (\mathcal{V} * \tilde{\mathbf{t}})_{\Gamma_{D,i}}(\mathbf{x}; t_m), \mathbf{w}(\mathbf{x}) \rangle_{\Gamma_{D,i}} = \int_0^{t_m} \int_{\Gamma_{D,i}} \mathbf{w}(\mathbf{x}) \cdot \int_{\Gamma_{D,i}} \mathbf{U}(\mathbf{y}-\mathbf{x}; t-\tau) \cdot \tilde{\mathbf{t}}(\mathbf{y}; \tau)\, \mathrm{d}s_{\mathbf{y}}\, \mathrm{d}s_{\mathbf{x}}\, \mathrm{d}\tau
$$

$$
\approx \sum_{k=0}^m \int_{\Gamma_{D,i}} \mathbf{w}(\mathbf{x}) \cdot \int_{\Gamma_{D,i}} \omega_{m-k}(\hat{\mathbf{U}}, \Delta t) \cdot \tilde{\mathbf{t}}(\mathbf{y}; k\Delta t)\, \mathrm{d}s_{\mathbf{y}}\, \mathrm{d}s_{\mathbf{x}}
\tag{19}
$$

with the integration weights

$$\omega_{m-k}(\hat{\mathbf{U}}, \Delta t) = \frac{1}{L} \sum_{\ell=0}^{L-1} \hat{\mathbf{U}}(\mathbf{y} - \mathbf{x}; \frac{\gamma(\zeta_\ell)}{\Delta t}) \zeta_\ell^{-(m-k)} \ . \tag{20}$$

Finally, inserting (20) into (19) and rearranging yields

$$\langle (\mathcal{V} * \tilde{\mathbf{t}})_{\Gamma_{D,i}}(\mathbf{x}; t_m), \mathbf{w}(\mathbf{x}) \rangle_{\Gamma_{D,i}} \approx \sum_{k=0}^{m} \langle \hat{\mathcal{V}}_{m-k} \tilde{\mathbf{t}}_k, \mathbf{w} \rangle_{\Gamma_{D,i}} \tag{21}$$

with

$$\hat{\mathcal{V}}_{m-k} \tilde{\mathbf{t}}_k := \sum_{\ell=0}^{L-1} \frac{\zeta_\ell^{-(m-k)}}{L} \int_{\Gamma_{D,i}} \hat{\mathbf{U}}(\mathbf{y} - \mathbf{x}; \frac{\gamma(\zeta_\ell)}{\Delta t}) \cdot \tilde{\mathbf{t}}(\mathbf{y}; k \Delta t) \, ds_{\mathbf{y}} \ . \tag{22}$$

Equation (21) indicates that the application of the CQM as a time-stepping procedure to the time-dependent variational form (14) results in an variational form very similar to a series of static analysis – except that the Laplace transformed fundamental solutions have to be evaluated for several complex parameters $\gamma(\zeta_\ell)/\Delta t$.

Applying the same technique as for the single layer potential in (21) to all other involved boundary integral operators yields the following time discretized variational formulation

$$\sum_{k=0}^{m} \begin{bmatrix} \langle \hat{\mathcal{V}}_{m-k} \tilde{\mathbf{t}}_k, \mathbf{w} \rangle_{\Gamma_{D,i}} - \langle \hat{\mathcal{K}}_{m-k} \tilde{\mathbf{u}}_k, \mathbf{w} \rangle_{\Gamma_{D,i}} \\ \langle \hat{\mathcal{K}}'_{m-k} \tilde{\mathbf{t}}_k, \mathbf{v} \rangle_{\Gamma_{N,i}} + \langle \hat{\mathcal{D}}_{m-k} \tilde{\mathbf{u}}_k, \mathbf{v} \rangle_{\Gamma_{N,i}} \end{bmatrix} = \sum_{k=0}^{m} \begin{bmatrix} \langle (\frac{1}{2}\hat{I} + \hat{\mathcal{K}})_{m-k} \tilde{\mathbf{g}}_{D_k} - \hat{\mathcal{V}}_{m-k} \tilde{\mathbf{g}}_{N_k}, \mathbf{w} \rangle_{\Gamma_{D,i}} \\ \langle (\frac{1}{2}\hat{I} - \hat{\mathcal{K}}')_{m-k} \tilde{\mathbf{g}}_{N_k} - \hat{\mathcal{D}}_{m-k} \tilde{\mathbf{g}}_{D_k}, \mathbf{v} \rangle_{\Gamma_{N,i}} \end{bmatrix} \ . \tag{23}$$

The above expression perfectly represents the well known structure of classical boundary integral equations formulated directly in time domain. This structure becomes immediately clear when the used subscripts in (23) are considered a bit more detailed. Obviously, they form a discrete convolution for the times t_k. To point out this structure more clearly, the Laplace transformed identity operator $\hat{I}$ is discussed briefly. This operator contains the Laplace transformed Delta distribution $\mathcal{L}\{\delta\}(s)$ which is known to be 1. Therefore, the integration weights ω_n can directly be computed by using (17)

$$\omega_n = \frac{R^{-n}}{2\pi} \int_0^{2\pi} e^{-in\varphi} \, d\varphi = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases} \ . \tag{24}$$

This shows that the identity operator in (23) contributes only to the first time step as in classical time domain boundary element formulations.

In 3-d calculations, another similarity to the classical time domain formulations is the possibility of introducing a cutoff of the sum $\sum_{k=0}^{m}(\cdots)$. This cutoff corresponds to the material's behavior. In contrast to a viscoelastic material an elastic material has no memory. Until the compression wave has not arrived at the location $\mathbf{x}$ all operators must be zero (*causality*) and they must be again zero when the shear wave has passed. Contrary to the classical time dependent boundary integral formulations where the cutoff is a consequence of the analytical time integration [32], here, one has to be content with an estimate for it [39]

$$\omega_n \approx \frac{\left(\frac{r_{\max}}{c_2 \Delta t}\right)^n}{n!} e^{-\frac{3}{2} \frac{r_{\max}}{c_2 \Delta t}} \ . \tag{25}$$

In the above equation, $r_{\max}$ is the maximum distance in the discretized body, i.e., the largest distance the shear wave associated with the velocity c_2 has to travel. Moreover, for this estimate a BDF 2 as underlying multistep method is assumed. The use of other multistep methods will, of course, lead to other estimations. Hence, an upper limit $\bar{n}$ for calculating the integration weights can be estimated so that for all $n > \bar{n}$ the integration weights can be neglected in relation to the weights $n < \bar{n}$. The use of a cutoff leads to a significant optimization due to the fact that it is dispensable to calculate and, of course, to store the system matrices for the time steps $t_k, k > \bar{n}$. So, instead of obtaining a system of lower triangular Toeplitz block matrices one ends up with a banded system just by replacing the sums in (23) by $\sum_{k=0}^{m}(\cdots) \rightarrow \sum_{k=\max(0,m-\bar{n})}^{m}(\cdots)$. More information about the cutoff may be found in [39, 21].

3.2 Spatial discretization

After the variational formulation (14) has been discretized in time the remaining step is the spatial discretization of (23). Due to the similarities between (23) and systems occurring in static analysis this procedure follows the same rules already known for those types of variational formulations.

First, a triangulation of the boundary $\Gamma = \cup_{k=1}^{n} \tau_k$ is introduced, i.e., the boundary Γ is the union of n boundary elements τ_k. Further, for each boundary element τ_k the unknown boundary data belong either to $\Gamma_{D,i}$ or to $\Gamma_{N,i}$ in each direction i. With respect to this triangulation the subspaces

$$
\begin{aligned}
S_{h,i}^{\alpha}(\Gamma_{D,i}) &= \operatorname{span}\{\varphi_{i,k}^{\alpha}\}_{k=1}^{\nu_i} \\
S_{h,i}^{\beta}(\Gamma_{N,i}) &= \operatorname{span}\{\varphi_{i,k}^{\beta}\}_{k=1}^{\upsilon_i}
\end{aligned}
\tag{26}
$$

are defined containing ν_i polynomial shape functions φ^{α} of order α and υ_i polynomials φ^{β} of order β. The subspaces' dimensions ν_i and υ_i correspond to the number of unknowns on $\Gamma_{D,i}$ and $\Gamma_{N,i}$. Therefore, the unknown Dirichlet data and the unknown Neumann data can be approximated by

$$
\begin{aligned}
\tilde{t}_{h,i}^{\alpha}(\mathbf{x};t) &= \sum_{k=1}^{\nu_i} t_{i,k}(t)\, \varphi_k^{\alpha}(\mathbf{x}) \\
\tilde{u}_{h,i}^{\beta}(\mathbf{x};t) &= \sum_{k=1}^{\upsilon_i} u_{i,k}(t)\, \varphi_k^{\beta}(\mathbf{x})\ .
\end{aligned}
\tag{27}
$$

In the following, Neumann data are approximated by constant functions and Dirichlet data by linear functions, i.e., $\alpha = 0$, $\tilde{\mathbf{t}}_h = \tilde{\mathbf{t}}_h^0$ and $\beta = 1$, $\tilde{\mathbf{u}}_h = \tilde{\mathbf{u}}_h^1$. Further, to obtain symmetric system-matrices the test-functions $\mathbf{w}(\mathbf{x}), \mathbf{v}(\mathbf{x})$ used in the variational formulation (23) are approximated by polynomials corresponding to the approximations for the Neumann and Dirichlet data, i.e., $w_{h,i}^0 \in S_{h,i}^0$ and $v_{h,i}^1 \in S_{h,i}^1$. Finally, the shape functions approximating the geometry are chosen of the same order as the Dirichlet data, i.e., the geometry is also approximated with linear polynomials.

3.3 System of linear equations

For a time index m and for the total numbers of degrees of freedom $\nu = \sum_{i=1}^{3} \nu_i$, $\upsilon = \sum_{i=1}^{3} \upsilon_i$ let

$$\mathsf{V}_m \in \mathbb{R}^{\nu \times \nu}, \quad \mathsf{K}_m \in \mathbb{R}^{\nu \times \upsilon}, \quad \mathsf{D}_m \in \mathbb{R}^{\upsilon \times \upsilon} \tag{28}$$

denote the discretized single layer, double layer, and hypersingular operator, respectively. Moreover, the unknown Cauchy data for the time step m is represented by the two vectors

$$\mathsf{t}_m \in \mathbb{R}^\nu, \quad \mathsf{u}_m \in \mathbb{R}^\upsilon . \tag{29}$$

Then, together with the stated time and spatial discretizations a system of linear equations is obtained which represents nicely the structure of the introduced time discretized variational form (23). This banded system is composed by the already mentioned lower triangular Toeplitz block matrices and, finally, reads as

$$\begin{bmatrix} \mathsf{V}_0 & -\mathsf{K}_0 \\ \mathsf{K}_0^\mathsf{T} & \mathsf{D}_0 \end{bmatrix} \cdot \begin{bmatrix} \mathsf{t}_m \\ \mathsf{u}_m \end{bmatrix} = \begin{bmatrix} \tilde{\mathsf{f}}_m^D \\ \tilde{\mathsf{f}}_m^N \end{bmatrix} \tag{30}$$

with the given right hand side

$$\begin{bmatrix} \tilde{\mathsf{f}}_m^D \\ \tilde{\mathsf{f}}_m^N \end{bmatrix} = \begin{bmatrix} \mathsf{f}_m^D \\ \mathsf{f}_m^N \end{bmatrix} - \sum_{k=k_0}^{m-1} \begin{bmatrix} \mathsf{V}_{m-k} \cdot \mathsf{t}_k - \mathsf{K}_{m-k} \cdot \mathsf{u}_k \\ \mathsf{K}_{m-k}^\mathsf{T} \cdot \mathsf{t}_k + \mathsf{D}_{m-k} \cdot \mathsf{u}_k \end{bmatrix} . \tag{31}$$

In (30) and (31), the usual structure of a BE time stepping technique is observed. The vector $[\mathsf{f}_m^D, \mathsf{f}_m^N]^\mathsf{T}$ is given by

$$\begin{bmatrix} \mathsf{f}_m^D \\ \mathsf{f}_m^N \end{bmatrix} = \begin{bmatrix} \bar{\mathsf{K}}_0 \cdot \mathsf{g}_m^D - \mathsf{V}_0 \cdot \tilde{\mathsf{g}}_m^N \\ \bar{\mathsf{K}}_0' \cdot \mathsf{g}_m^N - \mathsf{D}_0 \cdot \tilde{\mathsf{g}}_m^D \end{bmatrix} + \sum_{k=k_0}^{m-1} \begin{bmatrix} \mathsf{K}_{m-k} \cdot \mathsf{g}_k^D - \mathsf{V}_{m-k} \cdot \tilde{\mathsf{g}}_k^N \\ \mathsf{K}_{m-k}' \cdot \mathsf{g}_k^N - \mathsf{D}_{m-k} \cdot \tilde{\mathsf{g}}_k^D \end{bmatrix} . \tag{32}$$

and contains the prescribed boundary data of the actual time step m whereas in the last term of (31) all information about the Cauchy data up to the previous time step $m-1$ is stored.

In (31) and (32), k_0 denotes the cutoff and is defined as $k_0 = \max(0, m-\bar{n})$. Moreover, in (32), the diagonal terms of the discrete double layer potential $\bar{\mathsf{K}}_0$ which correspond to the first time step are given by $\bar{\mathsf{K}}_0 = \frac{1}{2}\mathsf{I} + \mathsf{K}_0$ and, analogously, by $\bar{\mathsf{K}}_0' = \frac{1}{2}\mathsf{I} - \mathsf{K}_0'$ for the discrete adjoint double layer potential.

From (30), it is obvious that the solution requires only the inversion of the matrix corresponding to the first time step. Therefore, the matrix V_0, which is symmetric as a result of the Galerkin discretization, is decomposed via a *Cholesky-factorization*. Afterwards, the *Schur-Complement-System* is computed by

$$\mathsf{S}_0 = \mathsf{K}_0^\mathsf{T} \mathsf{V}_0^{-1} \mathsf{K}_0 + \mathsf{D}_0 . \tag{33}$$

Due to the symmetry of V_0 and D_0 the Schur-Complement S_0 is also symmetric and can be decomposed itself by a Cholesky-factorization.

Hence, the displacement field u and the tractions t can be found by solving

$$\mathsf{S}_0 \mathsf{u}_m = \tilde{\mathsf{f}}_m^N - \mathsf{K}_0^\mathsf{T} \mathsf{V}_0^{-1} \tilde{\mathsf{f}}_m^D \tag{34}$$

and

$$\mathsf{t}_m = \mathsf{V}_0^{-1} \left(\tilde{\mathsf{f}}_m^D + \mathsf{K}_0 \mathsf{u}_m \right) \tag{35}$$

for every time step $m = 0, \ldots, M-1$.

4 REGULARIZATION OF THE HYPERSINGULAR BILINEAR FORM

Since the fundamental solutions are used in Laplace domain (see section 3.1), only the Laplace transformed fundamental solutions are considered when speaking about elastodynamics from now on. Therefore, also the $\hat{(\,)}$ denoting the Laplace transformed fundamental solution will be omitted. Moreover, for sake of simplicity the functions' argument list is skipped whenever the meaning is clear by the functions' name themselves.

In elastostatics there exists a regularized representation of the hypersingular bilinear form which has been deduced by Han [24] who has used some basic results from elasticity given in [26]. His proof leads to a bilinear form of the hypersingular operator containing at least a weakly singular integral kernel which can be treated numerically by using some appropriate cubature rules. Here, this technique will be transfered to elastodynamics. Contrary to the regularization given by Becache et. al. [2] this formulation is deduced only for the isotropic case but features the benefit of being more efficient for a numerical treatment since the resulting bilinear form exhibits a much simpler structure.

A comment must be made concerning the topology of the body being under consideration. According to all other mentioned regularizations [24, 34, 36, 2] the body is assumed to be either closed or to be the unbounded complement of a closed domain. This assumption simplifies the application of Stokes theorem, since in this case the surface's boundary $\partial\Gamma$ vanishes and $\partial\Gamma = \{\emptyset\}$ holds. Further, bearing in mind that the involved integral kernels fulfill Sommerfeld's radiation condition [42], the surface's boundary integrals vanish also in the case of bodies with infinite expansion, e.g., elastic halfspaces. Therefore, the present regularization holds in principle also for these special problems. Nevertheless, problems might occur on a discrete level if a semi-infinite geometry like an elastic halfspace is considered. There, it is a common practice to model just a truncated part of the infinite geometry. Unfortunately, the emerging truncation's borderline represents the surface's boundary such that $\partial\Gamma \neq \{\emptyset\}$ holds. In the typical case of prescribed boundary data of Neumann type on $\partial\Gamma$, then, the regularization is just incomplete and, therefore, any numerical study will fail. In opposite for prescribed homogeneous Dirichlet data on $\partial\Gamma$, like it is common in crack analysis, the regularization will work perfectly since in this case the boundary part Γ_N on which the hypersingular operator has to be evaluated is closed.

Now, to adapt Han's proof to elastodynamics, first, the similarities between both fundamental solutions have to be worked out. These fundamental solutions are well known and go back to the works of Lord Kelvin (1848) and G. Stokes (1849). For instance, the elastostatic fundamental solution may be found in [27] whereas the elastodynamic case is treated in [11, 12].

The elastostatic fundamental solution can be written as

$$\mathbf{U}^{ES} = \frac{1}{\mu}\left[\Delta\chi^{ES}\mathbf{I} - \frac{\lambda+\mu}{\lambda+2\mu}\nabla_{\mathbf{y}}\nabla_{\mathbf{y}}\chi^{ES}\right], \quad \chi^{ES} = \frac{r}{8\pi}\,. \tag{36}$$

The respective representation in elastodynamics is

$$\mathbf{U}^{ED} = \frac{1}{\mu}\left[\Delta\chi^{ED}\mathbf{I} - \frac{\lambda+\mu}{\lambda+2\mu}\nabla_{\mathbf{y}}\nabla_{\mathbf{y}}\chi^{ED}\right] - \frac{1}{\mu}\kappa_1^2\chi^{ED}\mathbf{I}, \quad \chi^{ED} = \frac{1}{4\pi r}\frac{1}{\kappa_1^2 - \kappa_2^2}\left(e^{-\kappa_1 r} - e^{-\kappa_2 r}\right)\,. \tag{37}$$

For later purpose it is important to state that the scalar functions χ^{ES} and χ^{ED} fulfill the partial differential equations

$$\Delta^2\chi^{ES} = -\delta(\mathbf{y}-\tilde{\mathbf{x}}) \tag{38}$$

$$(\Delta-\kappa_1^2)(\Delta-\kappa_2^2)\chi^{ED} = -\delta(\mathbf{y}-\tilde{\mathbf{x}}) \ . \tag{39}$$

In (36) and (37), as well as in the following, r denotes the Euclidean distance between the points $\mathbf{y}\in\Gamma$ and $\tilde{\mathbf{x}}\in\Omega$, i.e., $r=|\mathbf{y}-\tilde{\mathbf{x}}|$ holds. Further, $\kappa_1(s)=s\sqrt{\rho/\lambda+2\mu}$, and $\kappa_2(s)=s\sqrt{\rho/\mu}$ denote the wave numbers corresponding to the compression wave and to the shear wave, respectively.

First of all, it can be stated that both fundamental solutions are compositions of some regular scalar functions χ^{ES} and χ^{ED}. The elastodynamic fundamental solution's (37) first term is weakly singular due to the involved second order derivatives, whereas the last term in (37) is regular. Moreover, having in mind that the hypersingular operator is achieved by applying the traction operator twice it can be concluded that the term $\kappa_i^2/\mu\chi^{ED}\mathbf{I}$ will become weakly singular. Hence, no special attention has to be paid to this expression during the regularization process. Further, the terms in brackets in (36) and (37) are the identical operator applied to different functions χ. This similar structure motivates a similar treatment of the regularization process. Now in doing so, a kernel function $\mathbf{U}(\chi)$ is thought depending on some arbitrary but differentiable function $\chi(r)$

$$\mathbf{U} = \frac{1}{\mu}\left[\Delta\chi\mathbf{I} - \frac{\lambda+\mu}{\lambda+2\mu}\nabla_{\mathbf{y}}\nabla_{\mathbf{y}}\chi\right] \ . \tag{40}$$

Let us consider the double layer potential for some differentiable function $\mathbf{u}(\mathbf{y})$

$$(\mathcal{K}\mathbf{u})(\tilde{\mathbf{x}}) := \int_\Gamma [\mathcal{T}(\partial_{\mathbf{y}},\mathbf{n}(\mathbf{y}))\mathbf{U}]^\mathsf{T}\cdot\mathbf{u}(\mathbf{y})\,\mathrm{d}s_{\mathbf{y}} \quad \tilde{\mathbf{x}}\in\Omega, \mathbf{y}\in\Gamma \ . \tag{41}$$

As mentioned in section 2.2 the trace operator $\mathcal{T}(\partial_{\mathbf{y}},\mathbf{n}(\mathbf{y}))$ represents the stress-strain relation based on Hooke's law. This matrix differential operator can be defined by using the Günter derivatives (59)

$$\mathcal{T}(\partial_{\mathbf{y}},\mathbf{n}(\mathbf{y}))(\cdot) = 2\mu\mathcal{M}(\partial_{\mathbf{y}},\mathbf{n}(\mathbf{y}))(\cdot) + (\lambda+2\mu)\mathbf{n}(\mathbf{y})\nabla_{\mathbf{y}}\cdot(\cdot) - \mu(\mathbf{n}(\mathbf{y})\times\mathrm{curl}(\cdot)) \ . \tag{42}$$

Inserting (42) into (41) and integrating by parts gives

$$(\mathcal{K}\mathbf{u})(\tilde{\mathbf{x}}) = 2\mu\int_\Gamma \mathbf{U}\cdot(\mathcal{M}(\partial_{\mathbf{y}},\mathbf{n}(\mathbf{y}))\mathbf{u})\,\mathrm{d}s_{\mathbf{y}} - \int_\Gamma \Delta\chi\,(\mathcal{M}(\partial_{\mathbf{y}},\mathbf{n}(\mathbf{y}))\mathbf{u})\,\mathrm{d}s_{\mathbf{y}} + \int_\Gamma \frac{\partial\Delta\chi}{\partial\mathbf{n}(\mathbf{y})}\mathbf{u}\,\mathrm{d}s_{\mathbf{y}} \ . \tag{43}$$

Equation (43) is very useful with regard to the limiting process $\Omega\ni\tilde{\mathbf{x}}\to\mathbf{x}\in\Gamma$ since the above equation will lead to a weakly singular form of the double layer potential. This could be used to avoid the computation of the Cauchy principal value integrals. If χ is substituted by χ^{ES} from (36) a weakly singular formulation of the elastostatic double layer potential $(\mathcal{K}^{ES}\mathbf{u})(\mathbf{x})$ is obtained which is already given in [26]. Substituting χ by χ^{ED} from (37) and subtracting the term

$\kappa_1^2/\mu \int_\Gamma \left[(\mathcal{T}(\partial_\mathbf{y}, \mathbf{n}(\mathbf{y})) \chi^{ED} \cdot \mathbf{I}) \right]^\top \cdot \mathbf{u} \, ds_\mathbf{y}$ yields a regularized double layer potential in elastodynamics

$$
(\mathcal{K}^{ED}\mathbf{u})(\mathbf{x}) = 2\mu \int_\Gamma \mathbf{U}(\chi^{ED}) \cdot (\mathcal{M}(\partial_\mathbf{y}, \mathbf{n}(\mathbf{y}))\mathbf{u}) \, ds_\mathbf{y} - \int_\Gamma \Delta\chi^{ED} \left(\mathcal{M}(\partial_\mathbf{y}, \mathbf{n}(\mathbf{y}))\mathbf{u} \right) ds_\mathbf{y} + \int_\Gamma \frac{\partial \Delta\chi^{ED}}{\partial \mathbf{n}(\mathbf{y})} \mathbf{u} \, ds_\mathbf{y}
$$
$$
- \int_\Gamma \frac{d\chi^{ED}}{dr} \left[\left(\kappa_2^2 - 2\kappa_1^2 \right) \nabla_\mathbf{y} r \otimes \mathbf{n}(\mathbf{y}) + \kappa_1^2 \mathbf{n}(\mathbf{y}) \otimes \nabla_\mathbf{y} r + \kappa_1^2 \frac{\partial r}{\partial \mathbf{n}(\mathbf{y})} \mathbf{I} \right] \cdot \mathbf{u} \, ds_\mathbf{y}
$$
(44)

valid for almost all $\mathbf{x}, \mathbf{y} \in \Gamma$. Note that the last term of (44) is regular since a Taylor series expansion of $\frac{d\chi^{ED}}{dr}$ yields

$$
\frac{d\chi^{ED}}{dr} = \frac{1}{8\pi} - \frac{\kappa_1^2 + \kappa_1\kappa_2 + \kappa_2^2}{12(\kappa_1 + \kappa_2)\pi} r + \mathcal{O}(r^2) .
$$
(45)

Now, to go ahead with the treatment of the hypersingular kernel, the next step is the application of the operator $\mathcal{T}(\partial_{\tilde{\mathbf{x}}}, \mathbf{n}(\mathbf{x}))$ to the double layer potential $(\mathcal{K}\mathbf{u})(\tilde{\mathbf{x}})$ from (43). This results in

$$
\mathcal{T}(\partial_{\tilde{\mathbf{x}}}, \mathbf{n}(\mathbf{x}))(\mathcal{K}\mathbf{u})(\tilde{\mathbf{x}}) = \mu \int_\Gamma \frac{\partial \Delta\chi}{\partial \mathbf{n}(\mathbf{y}) \partial \mathbf{n}(\mathbf{x})} \mathbf{u} \, ds_\mathbf{y}
$$
$$
+ \int_\Gamma \left[\mathcal{M}(\partial_{\tilde{\mathbf{x}}}, \mathbf{n}(\mathbf{x})) \left(4\mu^2 \mathbf{U} - 3\mu\Delta\chi \mathbf{I} \right) \right] \cdot (\mathcal{M}(\partial_\mathbf{y}, \mathbf{n}(\mathbf{y}))\mathbf{u}) \, ds_\mathbf{y}
$$
$$
+ \psi_1 + \psi_2
$$
(46)

with

$$
\psi_1 := (\lambda + \mu) \mathbf{n}(\mathbf{x}) \int_\Gamma \left[\left(\nabla_{\tilde{\mathbf{x}}} \frac{\partial \Delta\chi}{\partial \mathbf{n}(\mathbf{y})} \right) \cdot \mathbf{u} - (\nabla_{\tilde{\mathbf{x}}} \Delta\chi) \cdot (\mathcal{M}(\partial_\mathbf{y}, \mathbf{n}(\mathbf{y}))\mathbf{u}) \right] ds_\mathbf{y}
$$
$$
\psi_2 := \mu \int_\Gamma \left[\left(\mathcal{M}(\partial_{\tilde{\mathbf{x}}}, \mathbf{n}(\mathbf{x})) \frac{\partial \Delta\chi}{\partial \mathbf{n}(\mathbf{y})} \right) \cdot \mathbf{u} + \frac{\partial \Delta\chi}{\partial \mathbf{n}(\mathbf{x})} (\mathcal{M}(\partial_\mathbf{y}, \mathbf{n}(\mathbf{y}))\mathbf{u}) \right] ds_\mathbf{y} .
$$
(47)

The above result can be obtained directly by following the proof given by Han [24]. For further simplification of (47) the identities (62) and (63) are needed. This yields

$$
\psi_1 = -(\lambda + \mu) \mathbf{n}(\mathbf{x}) \int_\Gamma \mathbf{n}(\mathbf{y}) \cdot \mathbf{u} \Delta^2\chi \, ds_\mathbf{y}
$$
$$
\psi_2 = -\mu \int_\Gamma \left[\mathcal{M}(\partial_{\tilde{\mathbf{x}}}, \mathbf{n}(\mathbf{x})) \Delta\chi \mathcal{M}(\partial_\mathbf{y}, \mathbf{n}(\mathbf{y})) \right]^\top \cdot \mathbf{u} \, ds_\mathbf{y} + \mu \mathcal{M}(\partial_{\tilde{\mathbf{x}}}, \mathbf{n}(\mathbf{x})) \int_\Gamma \Delta\chi \left(\mathcal{M}(\partial_\mathbf{y}, \mathbf{n}(\mathbf{y}))\mathbf{u} \right) ds_\mathbf{y}
$$
$$
- \mu \int_\Gamma \Delta^2\chi \left(\mathbf{n}(\mathbf{y}) \otimes \mathbf{n}(\mathbf{x}) - \mathbf{n}(\mathbf{x}) \otimes \mathbf{n}(\mathbf{y}) \right) \cdot \mathbf{u} \, ds_\mathbf{y} .
$$
(48)

Now it is time to pick up the pieces. Inserting the expressions ψ_1 and ψ_2 from (48) into (46)

13

and using (61) finally gives

$$
\begin{aligned}
\mathcal{T}(\partial_{\tilde{\mathbf{x}}},\mathbf{n}(\mathbf{x}))(\mathcal{K}\mathbf{u})(\tilde{\mathbf{x}}) = {}& \mu \sum_{k=1}^{3} \frac{\partial}{\partial S_k(\mathbf{x})} \int_{\Gamma} \Delta\chi \frac{\partial}{\partial S_k(\partial_{\mathbf{y}},\mathbf{n}(\mathbf{y}))} \mathbf{u}\, \mathrm{d}s_{\mathbf{y}} \\
& + \mathcal{M}(\partial_{\tilde{\mathbf{x}}},\mathbf{n}(\mathbf{x})) \int_{\Gamma} \left(4\mu^2\mathbf{U} - 2\mu\Delta\chi\mathbf{I}\right) \cdot (\mathcal{M}(\partial_{\mathbf{y}},\mathbf{n}(\mathbf{y}))\mathbf{u})\, \mathrm{d}s_{\mathbf{y}} \\
& - \mu \int_{\Gamma} \left[\mathcal{M}(\partial_{\tilde{\mathbf{x}}},\mathbf{n}(\mathbf{x}))\Delta\chi\mathcal{M}(\partial_{\mathbf{y}},\mathbf{n}(\mathbf{y}))\right]^{\mathsf{T}} \cdot \mathbf{u}\, \mathrm{d}s_{\mathbf{y}} \\
& - \int_{\Gamma} \Delta^2\chi \left[\lambda(\mathbf{n}(\mathbf{y})\cdot\mathbf{u})\mathbf{n}(\mathbf{x}) + \mu(\mathbf{n}(\mathbf{x})\cdot\mathbf{u})\mathbf{n}(\mathbf{y}) + \mu(\mathbf{n}(\mathbf{y})\cdot\mathbf{n}(\mathbf{x}))\mathbf{u}\right] \mathrm{d}s_{\mathbf{y}}\,.
\end{aligned}
\tag{49}
$$

Remember that up to this point the above statement holds only for $\tilde{\mathbf{x}} \notin \Gamma$. Therefore, the next step will be the limiting process $\Omega \ni \tilde{\mathbf{x}} \to \mathbf{x} \in \Gamma$. To do so, the partial differential equations (38) and (39) have to be taken into account. As stated at the beginning of this section the function χ depends on the problem which is under consideration. In the elastodynamic case $\Delta^2\chi$ fulfills (39), i.e.,

$$
\Delta^2\chi^{ED} = \left(\kappa_1^2 + \kappa_2^2\right)\Delta\chi^{ED} - \kappa_1^2\kappa_2^2\chi^{ED}
\tag{50}
$$

due to the fact that the point $\tilde{\mathbf{x}}$ lies inside the domain and, therefore, $\delta(r) = 0$ holds. Substituting $\mathbf{U}$ by $\mathbf{U}^{ED}$, inserting (50) into (49), and taking the fundamental solution's last term from (37) into account yields the elastodynamic hypersingular operator

$$
(\mathcal{D}^{ED}\mathbf{u})(\tilde{\mathbf{x}}) = -\left(\mathcal{T}(\partial_{\tilde{\mathbf{x}}},\mathbf{n}(\mathbf{x}))(\mathcal{K}\mathbf{u})(\tilde{\mathbf{x}}) - \frac{\kappa_1^2}{\mu}\mathcal{T}(\partial_{\tilde{\mathbf{x}}},\mathbf{n}(\mathbf{x}))\int_{\Gamma}\left[\mathcal{T}(\partial_{\mathbf{y}},\mathbf{n}(\mathbf{y}))\chi^{ED}\mathbf{I}\right]^{\mathsf{T}}\cdot\mathbf{u}\,\mathrm{d}s_{\mathbf{y}}\right).
\tag{51}
$$

Note that also the elastostatic hypersingular operator as given by Han [24] can easily be obtained from (49). In this case $\mathbf{U}$ has to be substituted with $\mathbf{U}^{ES}$. Moreover, $\Delta^2\chi$ fulfills (38) and, therefore, $\Delta^2\chi^{ES} = 0$ holds due to the same reasons as mentioned for $\Delta^2\chi^{ED}$.

Now, it is almost done. Equation (51) contains no more hypersingular kernel functions. Thus, the limiting process $\Omega \ni \tilde{\mathbf{x}} \to \mathbf{x} \in \Gamma$ can be performed. The hypersingular bilinear form then follows by testing (51) with a differentiable function $\mathbf{v}(\mathbf{x})$. Afterwards, Stokes theorem is applied

a last time and the complete regularized bilinear form is obtained

$$
\begin{aligned}
\langle \mathcal{D}^{ED}\mathbf{u},\mathbf{v}\rangle &= \int_{\Gamma} \mathbf{v}(\mathbf{x}) \cdot \lim_{\Omega \ni \tilde{\mathbf{x}} \to \mathbf{x} \in \Gamma} (\mathcal{D}^{ED}\mathbf{u})(\tilde{\mathbf{x}})\, \mathrm{d}s_{\mathbf{x}} \\
&= \mu \int_{\Gamma}\int_{\Gamma} \Delta\chi^{ED} \left(\sum_{k=1}^{3} \frac{\partial}{\partial S_k(\partial_{\mathbf{x}},\mathbf{n}(\mathbf{x}))} \mathbf{v} \cdot \frac{\partial}{\partial S_k(\partial_{\mathbf{y}},\mathbf{n}(\mathbf{y}))} \mathbf{u} \right) \mathrm{d}s_{\mathbf{y}}\,\mathrm{d}s_{\mathbf{x}} \\
&\quad + \mu \int_{\Gamma}\int_{\Gamma} (\mathcal{M}(\partial_{\mathbf{x}},\mathbf{n}(\mathbf{x}))\mathbf{v})^{\mathsf{T}} \cdot \left(2\Delta\chi^{ED}\mathbf{I} - 4\mu\mathbf{U}(\chi^{ED})\right) \cdot (\mathcal{M}(\partial_{\mathbf{y}},\mathbf{n}(\mathbf{y}))\mathbf{u})\, \mathrm{d}s_{\mathbf{y}}\,\mathrm{d}s_{\mathbf{x}} \\
&\quad + \mu \int_{\Gamma}\int_{\Gamma} \sum_{j,\ell,k=1}^{3} \left(\mathcal{M}_{jk}(\partial_{\mathbf{x}},\mathbf{n}(\mathbf{x}))v_\ell\right) \Delta\chi^{ED} \left(\mathcal{M}_{j\ell}(\partial_{\mathbf{y}},\mathbf{n}(\mathbf{y}))u_k\right) \mathrm{d}s_{\mathbf{y}}\,\mathrm{d}s_{\mathbf{x}} \\
&\quad + \mu \int_{\Gamma}\int_{\Gamma} \mathbf{n}(\mathbf{x})\cdot\mathbf{v} \left(\kappa_2^2 - 2\kappa_1^2\right)\left(3\Delta\chi^{ED} - \kappa_2^2\chi^{ED}\right) \mathbf{n}(\mathbf{y})\cdot\mathbf{u}\, \mathrm{d}s_{\mathbf{y}}\,\mathrm{d}s_{\mathbf{x}} \\
&\quad + \mu \int_{\Gamma}\int_{\Gamma} \mathbf{n}(\mathbf{x})\cdot\mathbf{n}(\mathbf{y})\mathbf{v} \cdot \left[\kappa_1^2\left(\nabla_{\tilde{\mathbf{x}}}\nabla_{\mathbf{y}}\chi^{ED}\right) + \Delta^2\chi^{ED}\mathbf{I}\right]\cdot\mathbf{u} + \kappa_1^2 \frac{\partial^2\chi^{ED}}{\partial\mathbf{n}(\mathbf{y})\partial\mathbf{n}(\mathbf{x})}\mathbf{u}\cdot\mathbf{v}\, \mathrm{d}s_{\mathbf{y}}\,\mathrm{d}s_{\mathbf{x}} \\
&\quad + \mu \int_{\Gamma}\int_{\Gamma} \mathbf{v}\cdot\left[\kappa_1^2\left(\nabla_{\tilde{\mathbf{x}}}\frac{\partial\chi^{ED}}{\partial\mathbf{n}(\mathbf{y})}\right) + \Delta^2\chi^{ED}\mathbf{n}(\mathbf{y})\right]\mathbf{n}(\mathbf{x})\cdot\mathbf{u} \\
&\quad\quad + 2\left(\kappa_2^2 - 2\kappa_1^2\right)\mathbf{v}\cdot\left(\nabla_{\mathbf{y}}\frac{\partial\chi^{ED}}{\partial\mathbf{n}(\mathbf{x})}\right)\mathbf{n}(\mathbf{y})\cdot\mathbf{u}\, \mathrm{d}s_{\mathbf{y}}\,\mathrm{d}s_{\mathbf{x}} \\
&\quad + \mu \int_{\Gamma}\int_{\Gamma} \kappa_1^2\mathbf{v}\cdot\mathbf{n}(\mathbf{y})\left(\nabla_{\mathbf{y}}\frac{\partial\chi^{ED}}{\partial\mathbf{n}(\mathbf{x})}\right)\cdot\mathbf{u} + 2\left(\kappa_2^2 - 2\kappa_1^2\right)\mathbf{v}\cdot\mathbf{n}(\mathbf{x})\left(\nabla_{\tilde{\mathbf{x}}}\frac{\partial\chi^{ED}}{\partial\mathbf{n}(\mathbf{y})}\right)\cdot\mathbf{u}\, \mathrm{d}s_{\mathbf{y}} \ .
\end{aligned}
$$

$$(52)$$

Equation (52) contains only weak singularities due to the fact that all involved second order derivatives of χ^{ED} are of order $\mathcal{O}(r^{-1})$. This could be proven by using the chain rule

$$
\nabla_{\mathbf{y}}\nabla_{\mathbf{x}}\chi^{ED} = \left(\frac{\mathrm{d}^2\chi^{ED}}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}\chi^{ED}}{\mathrm{d}r}\right)\nabla_{\mathbf{y}}r\otimes\nabla_{\mathbf{x}}r - \frac{1}{r}\frac{\mathrm{d}\chi^{ED}}{\mathrm{d}r}\mathbf{I} =: \phi_1\nabla_{\mathbf{y}}r\otimes\nabla_{\mathbf{x}}r - \phi_2\mathbf{I} \ . \qquad (53)
$$

A series expansion of ϕ_1 and ϕ_2 then yields

$$
\begin{aligned}
\phi_1 &= -\frac{1}{8\pi r} + \mathcal{O}(r) \\
\phi_2 &= \frac{1}{8\pi r} - \frac{\kappa_1^2 + \kappa_1\kappa_2 + \kappa_2^2}{12\pi(\kappa_1 + \kappa_2)} + \mathcal{O}(r) \ .
\end{aligned}
\qquad (54)
$$

Hence, the representation (52) of the hypersingular bilinear form is obviously suitable for the numerical treatment. Here, all implementations are done by using cubature rules developed by Erichsen and Sauter [15, 37].

15

5 NUMERICAL EXAMPLES

In order to validate the proposed boundary element formulation, a 3-d elastodynamic rod with length $\ell_1 = 3$m, width $\ell_2 = 1$m, and height $\ell_3 = 1$m is considered as depicted in Fig. 1. The
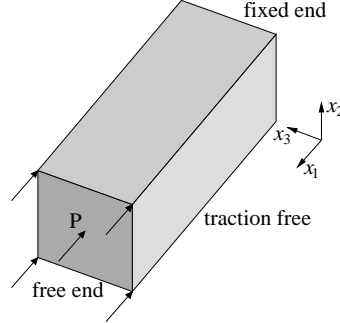

Figure 1: System and loading

rod is fixed on one end and excited by a pressure jump $t_1 = -1H(t)\,\mathrm{N/m^2}$ according to a unit step function $H(t)$ on the other free end. The remaining surfaces are traction free. The material data represent steel with Lamé's constants $\lambda = 0.0\,\mathrm{N/m^2}$, $\mu = 1.05 \cdot 10^{11}\,\mathrm{N/m^2}$, and the density $\rho = 7850\,\mathrm{kg/m^3}$. Note that $\lambda = 0.0\,\mathrm{N/m^2}$ is equivalent to an artificial Poisson's ratio of $\nu = 0$ instead of $\nu = 0.3$. The choice of this artificial value is used to be able to compare the results with a 1-d analytical solution of longitudinal waves in an elastodynamic column [17].

The Figures 2 & 3 shows a total of four regular meshes for the described model problem. While the first two meshes represents BEM discretizations the two remaining meshes are discretizations being used with the Finite Element Method (FEM). In the following, the BEM meshes will be referred to as BEM1 which is made up by 112 elements and 58 nodes and BEM2 which consists of 700 elements with 352 nodes. Analogously the FEM discretizations are titled as FEM1 made up of 24 elements and 63 nodes and FEM2 being composed of 375 elements and 576 nodes. As mentioned in section 3.2, for the BEM formulation the tractions are approximated with constant functions whereas the displacements are approximated linear. The FEM studies are done by using the classical Newmark algorithm ($\beta = 0.25, \gamma = 0.50$) [35]. This time stepping scheme is embedded in displacement-based FEM implementation using trilinear approximations for the Dirichlet data.

In order to compare some results for different time and spatial discretizations the dimensionless value

$$\beta = \frac{c_1 \Delta t}{h} \tag{55}$$

is introduced which can be referred to as the *Courant-Friedrichs-Lewy (CFL)* number [10]. Beside the velocity $c_1 = \sqrt{\lambda + 2\mu/\rho}$ of the compression wave the value β depends on the time step size Δt and the characteristic mesh size h. In 3-d, the definition of the mesh size h is not unique since the elements are 2-dimensional surfaces. Here, the triangles' cathetus is used for h, i.e., $h = 0.5$ m for mesh BEM1, and $h = 0.2$ m for mesh BEM2, respectively. The mesh sizes as well as the time step sizes of the FEM discretizations are in accordance to the BEM discretizations.
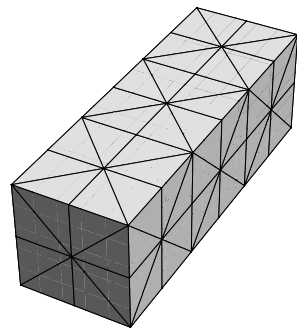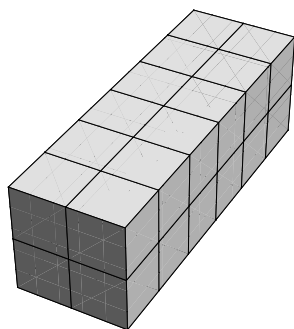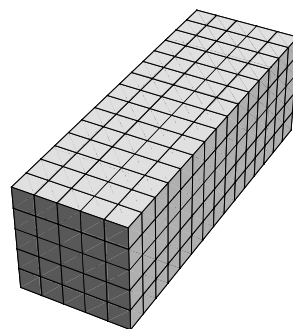
(a) BEM1: 112 elements, 58 nodes
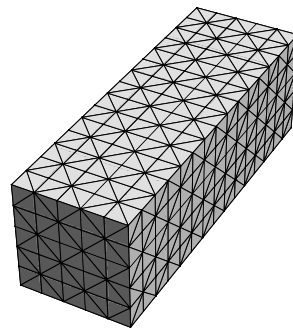
(b) BEM2: 700 elements, 352 nodes

Figure 2: Spatial discretizations (BEM) of the considered bar

(a) FEM1: 24 elements, 63 nodes

(b) FEM2: 375 elements, 576 nodes

Figure 3: Spatial discretizations (FEM) of the considered bar

Figure 4 depicts the longitudinal displacements at the point *P* for mesh BEM1 and mesh BEM2 with respect to different time step sizes. As expected, an increasing time step size leads to a larger numerical damping for both meshes. When considering the results for the finer grid BEM2, the largest CFL number yields almost the same results as the finest time discretization. This observation confirms the experience with CQM based BEM formulations that a more exact spatial discretization yields a more stable time stepping procedure.

Concerning the choice of the CFL number, several numerical studies have shown the existence of a lower bound for this number. As depicted in Fig. 4, for the present boundary element formulation the time stepping scheme gets unstable for a CFL number of $\beta \approx 0.04$. For clarity of the pictures, the results for this unstable time step size is shown only in Fig. 4(b). In contrast to that lower bound, there exist no upper bound for β. Even if β is chosen to be larger than one the results are still stable but the numerical damping increases such that the dynamic solution approaches the static solution.

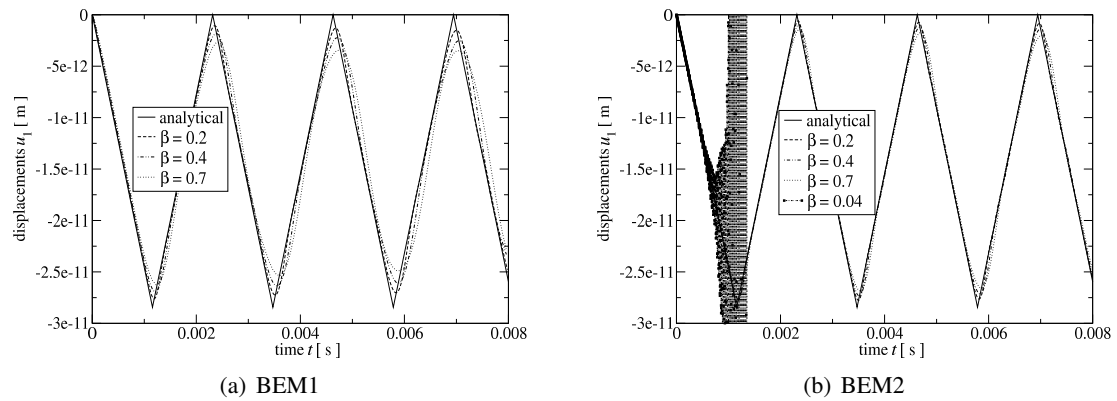(a) BEM1   (b) BEM2

Figure 4: Longitudinal displacements at the free end versus time: Different Δt

The traction solutions confirm the already stated observations: This solution is given for a point at the fixed end opposite to the point *P* and is shown for both meshes in Fig. 5. For mesh BEM1 as well as for mesh BEM2 a CFL number of $\beta = 0.2$ leads to the best results.

Two FEM results as depicted in Fig. 6 are given for completeness. Using the same mesh sizes for both, the BEM and the FEM meshes, it can be stated that the results of the BEM solution are very smooth while the FEM solutions are a bit more rough. Especially for the coarse grid solution depicted in Fig. 6(b) this roughness is very conspicuous. Another difference between both numerical methods occurs due to the different time stepping procedures. While both time stepping schemes are very stable, there is some overtone in the Newmark algorithm. This overtone is characteristic for the chosen Newmark algorithm and would become even more visible if a longer time period would be under consideration. In contrast, the CQM scheme exhibits minor damping.

Finally, a comparison between the CQM based collocation method as proposed in [39] and the actual presented symmetric Galerkin method reflects clearly the advantage of the latter method. In Figs. 7 and 8 long time solutions are depicted with respect to the coarse mesh BEM1. Moreover, for both simulations a CFL number of $\beta = 0.2$ is chosen. At first, the displacement solution
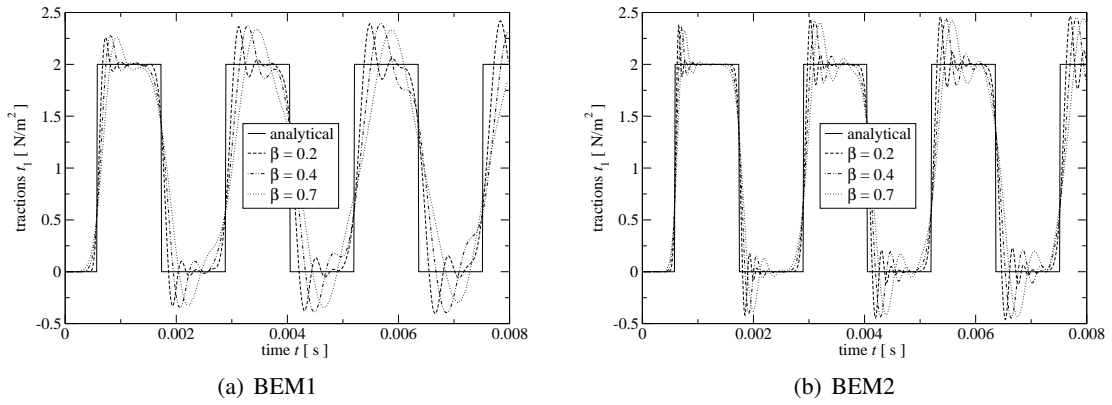
(a) BEM1

(b) BEM2

Figure 5: Normal traction at the fixed end versus time: Different Δt

(a) BEM2 vs. FEM2
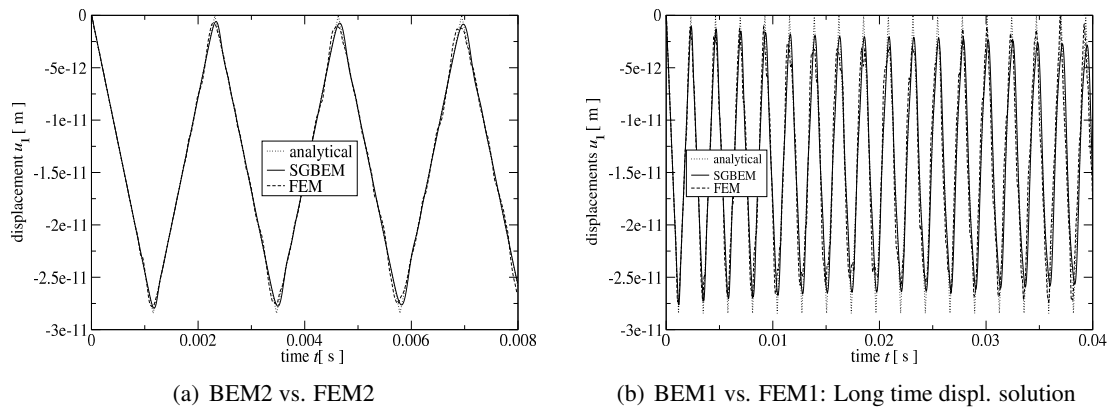
(b) BEM1 vs. FEM1: Long time displ. solution

Figure 6: Comparison of BEM and FEM displacement solutions

19

in Fig. 7 exhibits much more numerical damping effects in the collocation case than it does for the symmetric Galerkin method. The collocation result has also a phase shift for large times which is not visible for the Galerkin formulation.
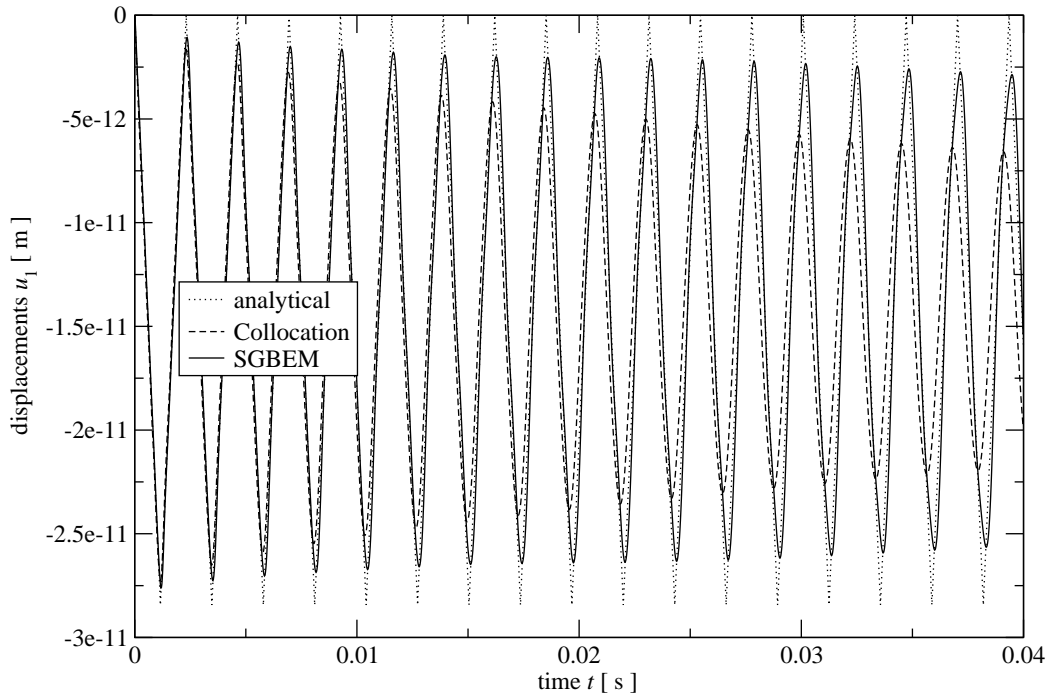

Figure 7: SGBEM vs. Collocation: Long time displacement solution for mesh BEM1

Besides the increasing damping for the displacement solution the collocation method reveals numerical instabilities with regard to the traction solution (see Fig. 8). This traction solution becomes completely instable and is therefore only depicted up to the time $t = 0.034$s. In contrast to this instability the Galerkin method is still stable during the whole observation time, although the numerical solution's quality decreases with increasing time.

6 CONCLUSIONS

A boundary element method for elastodynamics based on a Galerkin discretization in space and on the Convolution Quadrature Method in time was presented. To obtain a symmetric, or to be more precise, a skew-symmetric formulation, also the usage of the second boundary integral equation is required. Since the usage of the second boundary integral equation demands the computation of hypersingular kernel functions, a regularization of the elastodynamic hypersingular integral operator was presented which leads to a weakly singular bilinear form. The regularization is based mainly on integrations by parts and transfers a proof given for elastostatics to elastodynamics. This is based on the similar structure of the elastostatic fundamental solution compared to the Laplace transformed elastodynamic fundamental solution. Moreover, during the whole regularization process the use of the elastostatic fundamental solutions was
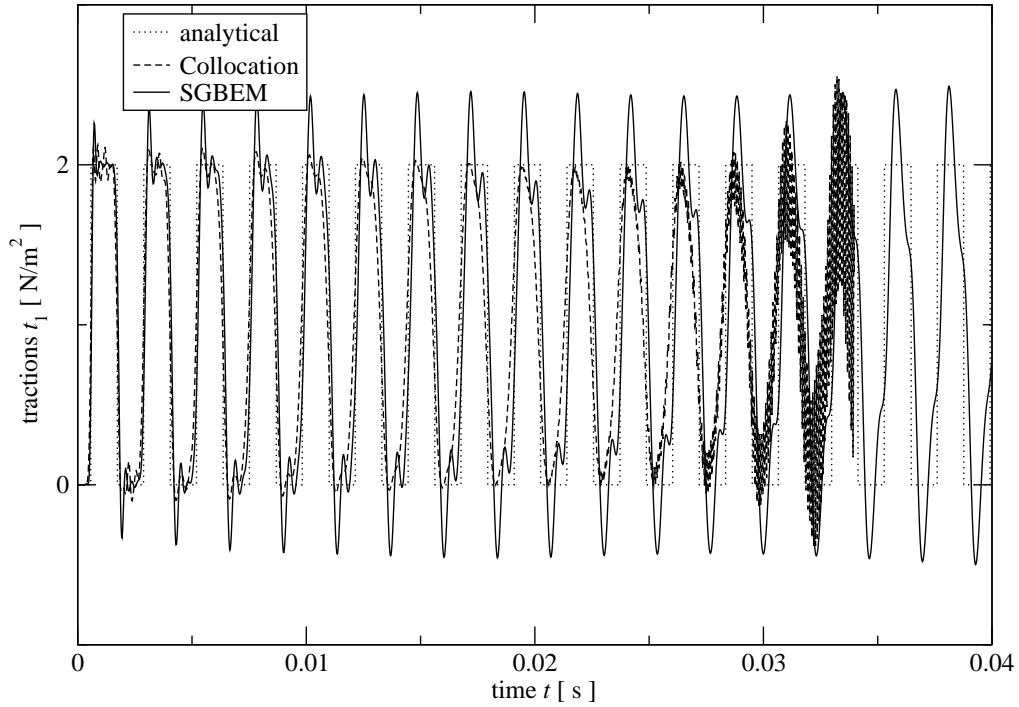
Figure 8: SGBEM vs. Collocation: Long time traction solution for mesh BEM1

not necessary like in other regularization strategies in dynamics. Hence, no difference terms between the static and the dynamic fundamental solutions have to be computed which avoids possible numerical problems.

The presented numerical examples show that this approach has obviously better numerical stability properties than the wider used collocation methods. Therefore, the positive results already obtained in statics with the symmetric Galerkin methods continues also in time domain analysis. Nevertheless, it has to be mentioned that due to the variational form the computation of the matrix entries is more costly. Hence, at the moment the overall computation is more time consuming than for the standard collocation method. In future, these computational costs should be reduced by, e.g, the application of some adaptive integration rules or the application of fast-methods like adaptive cross approximation (ACA) [1] or $\mathcal{H}$-matrices [22].

Moreover, one of the biggest advantages of boundary element methods in time domain, namely the relative easy feasibility of modelling wave propagation phenomena within semi-infinite domains, e.g., half-spaces, was lost since the whole regularization process assumes either a closed surface or vanishing integrals over the surface's boundary. Although this is true for non discretized systems it may not hold for the respective discretized problem. To overcome this drawback special techniques have to be worked out which will be presented in a forthcoming paper.

A TANGENTIAL SURFACE DERIVATIVES

A.1 Definitions and properties

The whole regularization process presented in section 4 is based on applications of some variants of Stokes theorem which should be recalled briefly. Let Γ be a surface with the outward unit normal vector $\mathbf{n}(\mathbf{y})$. Further, let $\partial\Gamma$ denote the surface's boundary and let $\mathbf{p}(\mathbf{y})$ be the unit tangent to $\partial\Gamma$. Then, the classical Stokes theorem for some differentiable vector field $\mathbf{v}(\mathbf{y})$ reads as

$$\int_{\Gamma} (\nabla_{\mathbf{y}} \times \mathbf{v}) \cdot \mathbf{n}(\mathbf{y})\, ds_{\mathbf{y}} = \int_{\partial\Gamma} \mathbf{v} \cdot \mathbf{p}(\mathbf{y})\, d\sigma_{\mathbf{y}} \,. \tag{56}$$

By introducing the *surface curl*

$$\frac{\partial}{\partial \mathbf{S}(\partial_{\mathbf{y}}, \mathbf{n}(\mathbf{y}))} = \mathbf{n}(\mathbf{y}) \times \nabla_{\mathbf{y}} \tag{57}$$

and assuming a closed surface Γ in the following, i.e., $\partial\Gamma = \{\emptyset\}$, Stokes theorem (56) can be written as

$$\int_{\Gamma} \frac{\partial}{\partial \mathbf{S}(\partial_{\mathbf{y}}, \mathbf{n}(\mathbf{y}))} \cdot \mathbf{v}\, ds_{\mathbf{y}} = 0 \,. \tag{58}$$

The introduced surface curl (57) is a so-called tangential differential operator since $\frac{\partial}{\partial \mathbf{S}(\partial_{\mathbf{y}}, \mathbf{n}(\mathbf{y}))} \cdot \mathbf{n}(\mathbf{y}) = 0$ holds. The operator's definition is taken from [26]. An analysis of the presented tangential differential operators may be also found in [14]. Here, a short review of their properties is given from a more engineering point of view. The used notation follows directly [24, 26]. Sometimes the surface curl (57) is also denoted by $\mathbf{curl}_{\Gamma}$ [34].

Moreover, the *Günter derivatives* $\mathcal{M}(\partial_{\mathbf{y}}, \mathbf{n}(\mathbf{y}))$ are used during the regularization process. These derivatives form an antisymmetric matrix operator [26, 19] and are defined as

$$\mathcal{M}_{ij}(\partial_{\mathbf{y}}, \mathbf{n}(\mathbf{y})) = n_j(\mathbf{y}) \frac{\partial}{\partial y_i} - n_i(\mathbf{y}) \frac{\partial}{\partial y_j} = \sum_{k=1}^{3} \varepsilon_{kji} \frac{\partial}{\partial S_k(\partial_{\mathbf{y}}, \mathbf{n}(\mathbf{y}))} \,. \tag{59}$$

Following Stokes theorem it is easy to verify that for a closed surface Γ the Günter derivatives fulfill the following equation

$$\int_{\Gamma} \mathbf{u} \cdot (\mathcal{M}(\partial_{\mathbf{y}}, \mathbf{n}(\mathbf{y}))\mathbf{v})\, ds_{\mathbf{y}} = \int_{\Gamma} \mathbf{v} \cdot (\mathcal{M}(\partial_{\mathbf{y}}, \mathbf{n}(\mathbf{y}))\mathbf{u})\, ds_{\mathbf{y}}, \quad \mathbf{u}(\mathbf{y}) \in \mathbb{R}^3, \mathbf{v}(\mathbf{y}) \in \mathbb{R}^3 \,. \tag{60}$$

Again, (60) holds only for a closed surface $\partial\Gamma$.

A.2 Useful equalities

In section 4, the expressions given in (46) and (47) are simplified by making use of the following identities. These identities are a more generalized version of identities already given in [24] and

can be obtained by some basic calculations

$$\frac{\partial^2 \Delta\chi}{\partial \mathbf{n}(\mathbf{y})\partial \mathbf{n}(\mathbf{x})} = -\frac{\partial}{\partial \mathbf{S}(\partial_{\mathbf{y}}, \mathbf{n}(\mathbf{y}))} \cdot \frac{\partial}{\partial \mathbf{S}(\partial_{\tilde{\mathbf{x}}}, \mathbf{n}(\mathbf{x}))} \Delta\chi \qquad (61)$$
$$- \mathbf{n}(\mathbf{y}) \cdot \mathbf{n}(\mathbf{x}) \Delta^2 \chi$$

$$\nabla_{\tilde{\mathbf{x}}}\left(\frac{\partial \Delta\chi}{\partial \mathbf{n}(\mathbf{y})}\right) = \mathcal{M}(\partial_{\mathbf{y}}, \mathbf{n}(\mathbf{y}))\left(\nabla_{\tilde{x}}\Delta\chi\right) - \mathbf{n}(\mathbf{y})\Delta^2\chi \qquad (62)$$

$$\mathcal{M}(\partial_{\tilde{\mathbf{x}}}, \mathbf{n}(\mathbf{x}))\left(\frac{\partial \Delta\chi}{\partial \mathbf{n}(\mathbf{y})}\right) - \mathcal{M}(\partial_{\mathbf{y}}, \mathbf{n}(\mathbf{y}))\left(\frac{\partial \Delta\chi}{\partial \mathbf{n}(\mathbf{x})}\right) = -(\mathbf{n}(\mathbf{y})\otimes\mathbf{n}(\mathbf{x}) - \mathbf{n}(\mathbf{x})\otimes\mathbf{n}(\mathbf{y}))\Delta^2\chi$$
$$+[\mathcal{M}(\partial_{\mathbf{y}}, \mathbf{n}(\mathbf{y}))\mathcal{M}(\partial_{\tilde{\mathbf{x}}}, \mathbf{n}(\mathbf{x})) - \mathcal{M}(\partial_{\tilde{\mathbf{x}}}, \mathbf{n}(\mathbf{x}))\mathcal{M}(\partial_{\mathbf{y}}, \mathbf{n}(\mathbf{y}))]\Delta\chi \;.$$
$$(63)$$

References

[1] M. Bebendorf and S. Rjasanow. Adaptive Low-Rank Approximation of Collocation Matrices. *Computing*, 70:1–24, 2003.

[2] E. Becache, J. C. Nedelec, and N. Nishimura. Regularization in 3D for anisotropic elastodynamic crack and obstacle problems. *Journal of Elasticity*, 31(1):25–46, 1993.

[3] D. E. Beskos. Boundary element methods in dynamic analysis. *Applied Mechanics Review*, 40(1):1–23, 1987.

[4] D. E. Beskos. Boundary element methods in dynamic analysis: Part II (1986-1996). *Applied Mechanics Review*, 50(3):149–197, 1997.

[5] M. Bonnet and H. D. Bui. Regularization of the Displacement and Traction BIE for 3D Elastodynamics Using Indirect Methods. In J. H. Kane, G. Maier, N. Tosaka, and S. N. Atluri, editors, *Advances in Boundary Element Techniques*, Springer series in computational mechanics, pages 1–29. Springer Verlag Berlin Heidelberg New York, 1993.

[6] M. Bonnet, G. Maier, and C. Polizzotto. Symmetric Galerkin boundary element methods. *Applied Mechanics Review*, 51(11):669–703, 1998.

[7] I. Chudinovich. Boundary Equations in Dynamic Problems of the Theory of Elasticity. *Acta Applicandae Mathematicae*, 65:169–183, 2001.

[8] M. Costabel. *Time-dependent problems with the boundary integral equation method*, volume 1 of *Encyclopedia of Computational Mechanics*, chapter 25. John Wiley & Sons, New York, Chister, Weinheim, 2005.

[9] M. Costabel and E. P. Stephan. Integral equations for transmission problems in linear elasticity. *Journal of Integral Equations and Applications*, 2:211–223, 1990.

[10] R. Courant, K. Friedrichs, and H. Lewy. Über die partiellen Differenzengleichungen der mathematischen Physik. *Mathematische Annalen*, 100:32–74, 1928.

[11] T. A. Cruse. A Direct Formulation and Numerical Solution of the General Transient Elas-todynamic Problem. II. *Journal of Mathematical Analysis and Applications*, 22:341–355, 1968.

[12] T. A. Cruse and F. J. Rizzo. A Direct Formulation and Numerical Solution of the General Transient Elastodynamic Problem. I. *Journal of Mathematical Analysis and Applications*, 22:244–259, 1968.

[13] J. Domínguez. *Boundary Elements in Dynamics*. Computational Mechanics Publications, Southampton, Boston, 1993.

[14] R. Duduchava. The Green formula and layer potentials. *Integral Equations and Operator Theory*, 41(2):127–178, 2001.

[15] S. Erichsen and S. A. Sauter. Efficient automatic quadrature in 3-d Galerkin BEM. *Computer Methods in Applied Mechanics and Engineering*, 157:215–224, 1998.

[16] L. Gaul, M. Kögl, and M. Wagner. *Boundary Element Methods for Engineers and Scientists*. Springer-Verlag Berlin Heidelberg, 2003.

[17] K. F. Graff. *Wave motions in elastic solids*. Dover Publications, 1991.

[18] M. Guiggiani. Formulation and numerical treatment of boundary integral equations with hypersingular kernels. In V. Sladek and J. Sladek, editors, *Singular Integrals in Boundary Element Methods*, pages 85–124. Computational Mechanics Publications, 1998.

[19] N. M. Günter. *Potential theory, and its applications to basic problems of mathematical physics*. Frederick Ungar Publishing, New York, 1967.

[20] T. Ha-Duong. On Retarded Potential Boundary Integral Equations and their Discretisation. In M. Ainsworth, P. Davies, D. Duncan, P. Martin, and B. Rynne, editors, *Topics in Computational Wave Propagation: Direct and Inverse Problems*, pages 301–336. Springer-Verlag Berlin, 2003.

[21] W. Hackbusch, W. Kress, and S. Sauter. Sparse Convolution Quadrature for Time Domain Boundary Integral Formulations of the Wave Equation by Cutoff and Panel-Clustering. In M. Schanz and O. Steinbach, editors, *Boundary Element Analysis*, volume 29 of *Lecture Notes in Applied and Computational Mechanics*, pages 113–134. Springer-Verlag Berlin Heidelberg, 2007.

[22] W. Hackbush. A Sparse Matrix Arithmetic Based on $\mathcal{H}$-Matrices. Part I: Introduction to $\mathcal{H}$-Matrices. *Computing*, 62:89–108, 1999.

[23] J. Hadamard. *Lectures on Cauchy's problem in linear partial differential equations*. Dover Publications, Inc. *(reprint)*, 1952.

[24] H. Han. The boundary integro-differential equations of three-dimensional Neumann problem in linear elasticity. *Numerische Mathematik*, 68:269–281, 1994.

[25] G. Krishnasamy, L. W. Schmerr, T. J. Rudolphi, and F. J. Rizzo. Hypersingular Boundary Integral Equations: Some Applications in Acoustic and Elastic Wave Scattering. *Journal of Applied Mechanics*, 57:404–414, 1990.

[26] V. D. Kupradze, editor. *Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity*, volume 25 of *Applied mathematics and mechanics*. North Holland, Amsterdam, 1979.

[27] A. E. H. Love. *A treatise on the mathematical theory of elasticity*, volume 1. Cambridge University Press, 1892.

[28] C. Lubich. Convolution quadrature and discretized operational calculus I. *Numerische Mathematik*, 52:129–145, 1988.

[29] C. Lubich. Convolution quadrature and discretized operational calculus II. *Numerische Mathematik*, 52:413–425, 1988.

[30] C. Lubich. On the multistep time discretization of linear initial-boundary value problems and their boundary integral equations. *Numerische Mathematik*, 67:365–389, 1994.

[31] C. Lubich and R. Schneider. Time discretization of parabolic boundary integral equations. *Numerische Mathematik*, 63:455–481, 1992.

[32] W. J. Mansur. *A Time-Stepping Technique to Solve Wave Propagation Problems Using the Boundary Element Method*. PhD thesis, University of Southampton, 1983.

[33] A. W. Maue. Zur Formulierung eines allgemeinen Beugungsproblems durch eine Integralgleichung. *Zeitschrift für Physik*, 126(7–9):601–618, 1949.

[34] J.C. Nedelec. Integral equations with non integrable kernels. *Integral Equations and Operator Theory*, 5:563–672, 1982.

[35] N. M. Newmark. A method of computation for structural dynamics. *ASCE Journal of the Engineering Mechanics Division*, 85:67–93, 1959.

[36] N. Nishimura and S. Kobayashi. A regularized boundary integral equation method for elastodynamic crack problems. *Computational Mechanics*, 4:319–328, 1989.

[37] S. Sauter and C. Schwab. *Randelementmethoden*. B.G. Teubner Verlag, Stuttgart Leipzig Wiesbaden, 2004.

[38] S. A. Sauter and C. Lage. Transformation of hypersingular integrals and black-box cubature. *Mathematics of Computation*, 70:223–250, 2001.

[39] M. Schanz. *Wave Propagation in Viscoelastic and Poroelastic Continua: A Boundary Element Approach*, volume 2 of *Lecture Notes in Applied Mechanics*. Springer-Verlag Berlin Heidelberg, 2001.

[40] M. Schanz and H. Antes. A new visco- and elastodynamic time domain Boundary Element formulation. *Computational Mechanics*, 20:452–459, 1997.

[41] S. Sirtori. General stress analysis method by means of integral equations and boundary elements. *Meccanica*, 14:210–218, 1979.

[42] A. Sommerfeld. *Partial Differential Equations in Physics*. Academic Press, New York, 1949.

[43] O. Steinbach. *Numerical Approximation Methods for Elliptic Boundary Value Problems*. Springer Science+Business Media, LLC, New York, 2008.