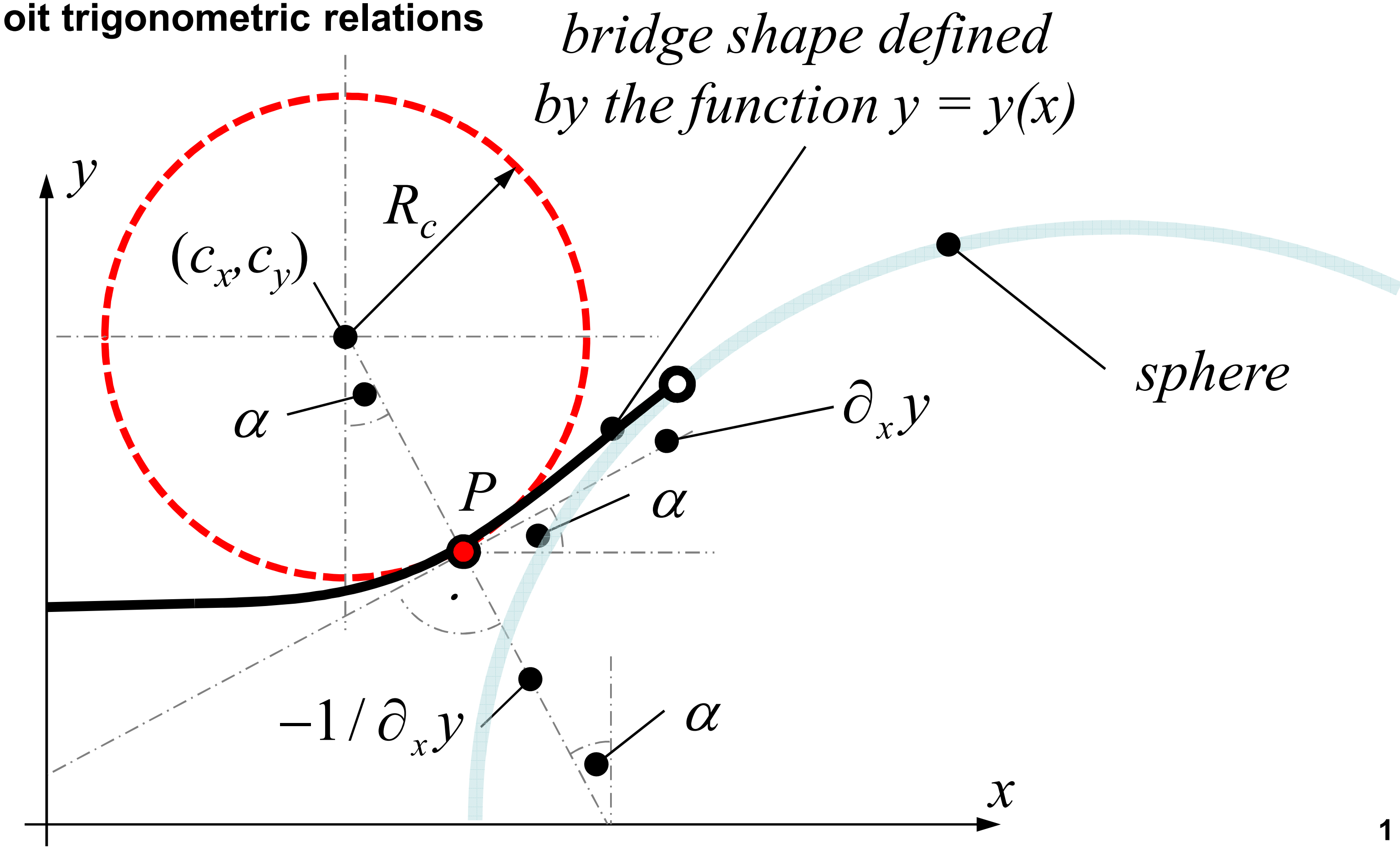
Derivation of the Young-Laplace Equation

- **Shape Description**
  - **Consider curve $y = y(x)$**
  - **Align circle with radius R_c to curve**
  - **Exploit trigonometric relations**



Derivation of the Young-Laplace Equation

- **Shape Description**
  - **The tangent is described by**

$$\tan\left(\alpha\right) = \partial_x y$$

  - **We observe the following for the vertical center position c_y of the circle**

$$\cos\left(\alpha\right) = \frac{c_y - y}{R_c}$$

  - **… and subsequent differentiation yields**

$$\partial_y \cos\left(\alpha\right) = -\frac{1}{R_c}$$

  - **Thus, an expression for the cosine of α will provide the radius of curvature**

Derivation of the Young-Laplace Equation

- **Shape Description**
  - **From basic trigonometric relationships we find that the tangent is related to the cosine via**

$$\cos^2(\alpha) = 1/\left(1+\tan^2(\alpha)\right)$$

  - **Thus, we get (where we choose a positive sign when taking the root)**

$$\cos(\alpha) = 1/\sqrt{1+[\partial_x y]^2}$$

  - **..and differentiation with respect to *y* yields**

$$\partial_y \cos(\alpha) = \frac{1}{\partial_x y}\partial_x\left[1+(\partial_x y)^2\right]^{-1/2}$$

$$\partial_y \cos(\alpha) = \frac{1}{\partial_x y}\frac{-1}{2}\left[1+(\partial_x y)^2\right]^{-3/2} 2\partial_x y\ \partial_{xx} y = \frac{-\partial_{xx} y}{\left[1+(\partial_x y)^2\right]^{3/2}}$$

Derivation of the Young-Laplace Equation

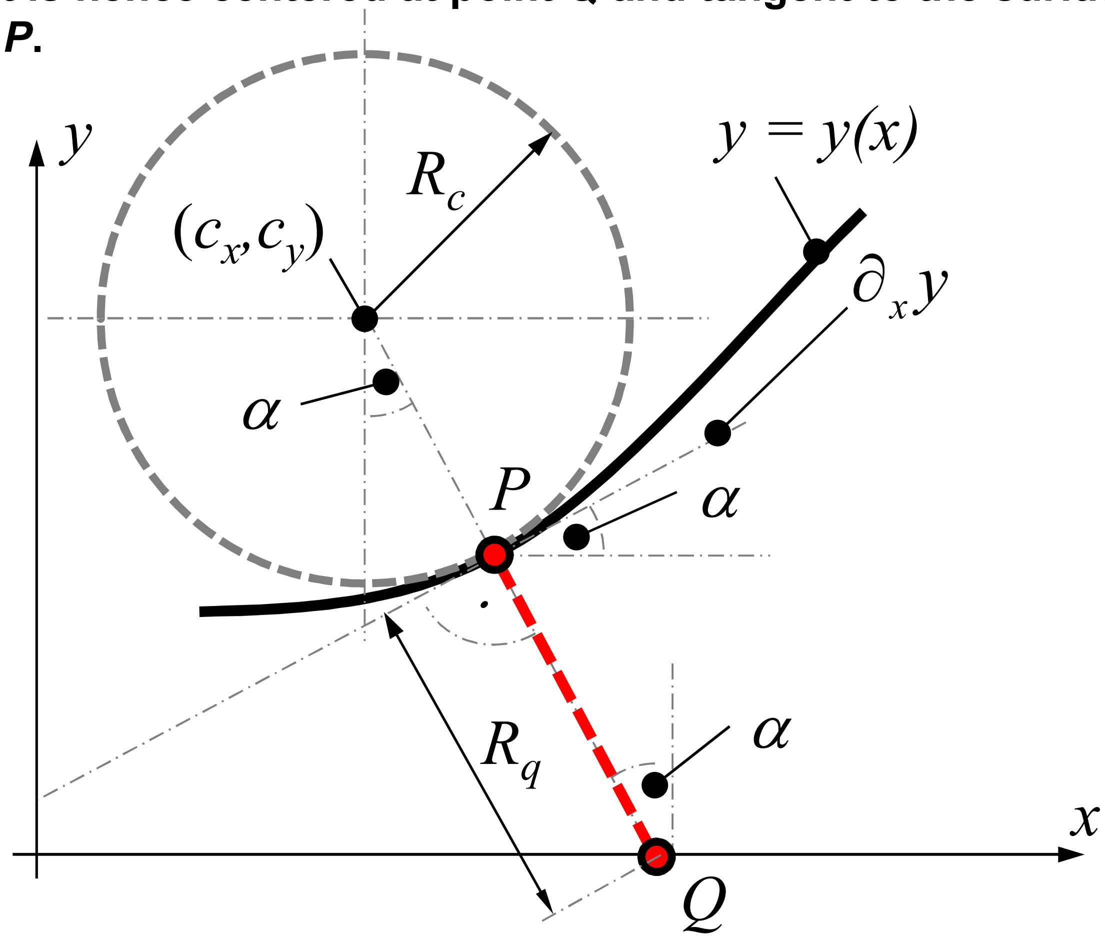
- **Shape Description**
  - **Thus, we finally arrive at**

$$-\partial_y \cos(\alpha) = \frac{1}{R_c} = \frac{\partial_{xx} y}{\left[1+(\partial_x y)^2\right]^{3/2}}$$

  - **The above expression provides and equation for the first principal radius of the bridge. In order to compute the curvature of the surface, we next derive an expression for the second principal radius that is arranged normal to the first radius.**

Derivation of the Young-Laplace Equation

- **Second Principal Radius**
  - **The second principal radius R_q is oriented normal to the first radius. It is hence centered at point Q and tangent to the surface at point P.**



Derivation of the Young-Laplace Equation

- **Second Principal Radius**
  - From basic trigonometric relations we observe that

$$\cos(\alpha) = y / R_q$$

  - Thus, we again exploit the basic trigonometric relationship

$$\cos^2(\alpha) = 1/\left(1+\tan^2(\alpha)\right)$$

  - …to arrive at (where we choose a negative sign when taking the root)

$$\frac{\cos(\alpha)}{y} = \frac{-1}{y\sqrt{1+[\partial_x y]^2}} = \frac{1}{R_q}$$

Derivation of the Young-Laplace Equation

- **The Curvature of the Liquid Bridge**
  - **For an arbitrarily shaped surface with the principal radii R_c and R_q, the curvature H is defined by**

$$H = \frac{1}{2}\left(\frac{1}{R_c} + \frac{1}{R_q}\right)$$

  - **Thus, the curvature of a liquid bridge at each position is defined via**

$$\boxed{2H = \frac{\partial_{xx} y}{\left[1+\left(\partial_x y\right)^2\right]^{3/2}} - \frac{1}{y\sqrt{1+\left[\partial_x y\right]^2}}}$$

Derivation of the Young-Laplace Equation

- **The Pressure in the Liquid Bridge**
  - **We now exploit the Young-Laplace equation, which in its most fundamental form reads**

$$\Delta p = 2\gamma H$$

  - **Thus, the pressure jump across a free surface is proportional to the curvature and the surface tension γ.**
  - **In case we assume no flow, a fixed ambient pressure in the gas surrounding the liquid bridge, as well as a negligible effect of gravity, the above pressure jump must be a constant. Thus, we ultimately arrive at the governing equation of the shape of a liquid bridge which reads**

$$2H = \frac{\Delta p}{\gamma} = \frac{\partial_{xx} y}{\left[1+\left(\partial_x y\right)^2\right]^{3/2}} - \frac{1}{y\sqrt{1+\left[\partial_x y\right]^2}}$$

Derivation of the Young-Laplace Equation

- **The final Governing Equation for the Liquid Bridge Shape**
  - **It is now illustrative to normalize this governing equation with the sphere radius a to arrive at a dimensionless version of the Young-Laplace equation for a liquid bridge**

$$\boxed{2H^{*} = 2H\,a = \frac{\Delta p\, a}{\gamma} = \frac{\partial_{XX} Y}{\left[1+\left(\partial_X Y\right)^2\right]^{3/2}} - \frac{1}{Y\sqrt{1+\left[\partial_X Y\right]^2}}}$$

  - **Here we have used the dimensionless position coordinates *X* and *Y*, which are defined by *X* = *x*/*y* and *Y* = *y*/*a*.**

Derivation of the Young-Laplace Equation

- **Simplified Governing Equation for the Liquid Bridge Shape**
  - **Often the second principal radius is much larger compared to the first one. Thus, one may estimate the capillary pressure following**

$$\Delta p \approx \frac{\gamma}{R_c} = \frac{\gamma}{a} \frac{\partial_{XX} Y}{\left[1+\left(\partial_X Y\right)^2\right]^{3/2}}$$

  - **As a further simplification, one could assume that**

$$\partial_X Y << 1$$

  - **…to arrive at a simple linear version of the Young-Laplace equation:**

$$\Delta p \approx \frac{\gamma}{R_c} = \frac{\gamma}{a} \partial_{XX} Y$$

Derivation of the Young-Laplace Equation

- **Boundary Conditions**
  - **To integrate the Young-Laplace equation in a straight forward fashion, two boundary conditions for *Y* and its first derivative at *X* = 0 must be specified. The initial condition for *Y* is set by the volume of the bridge and the separation distance (i.e., which is not known a priori, but can be calculated iteratively after integration of the bridge volume). For a liquid bridge between two equally-shaped particles the initial condition for the first derivative of Y is given by the symmetry condition, i.e.,**

$$\partial_X Y\Big|_{X=0} = 0$$

Derivation of the Young-Laplace Equation

- **Force Scaling and Force Estimate**
  - **The analysis of the liquid bridge shape shows that the (suction) pressure in the liquid bridge scales with**

$$\Delta p \propto \frac{\gamma}{a}$$

  - **The cross sectional area of the liquid bridge must scale with the radius of the spheres squared times π. Thus, we arrive at the principal scaling of the liquid bridge force**

$$F_{liq} \propto \Delta p \, \pi a^2 \propto \pi \gamma a$$

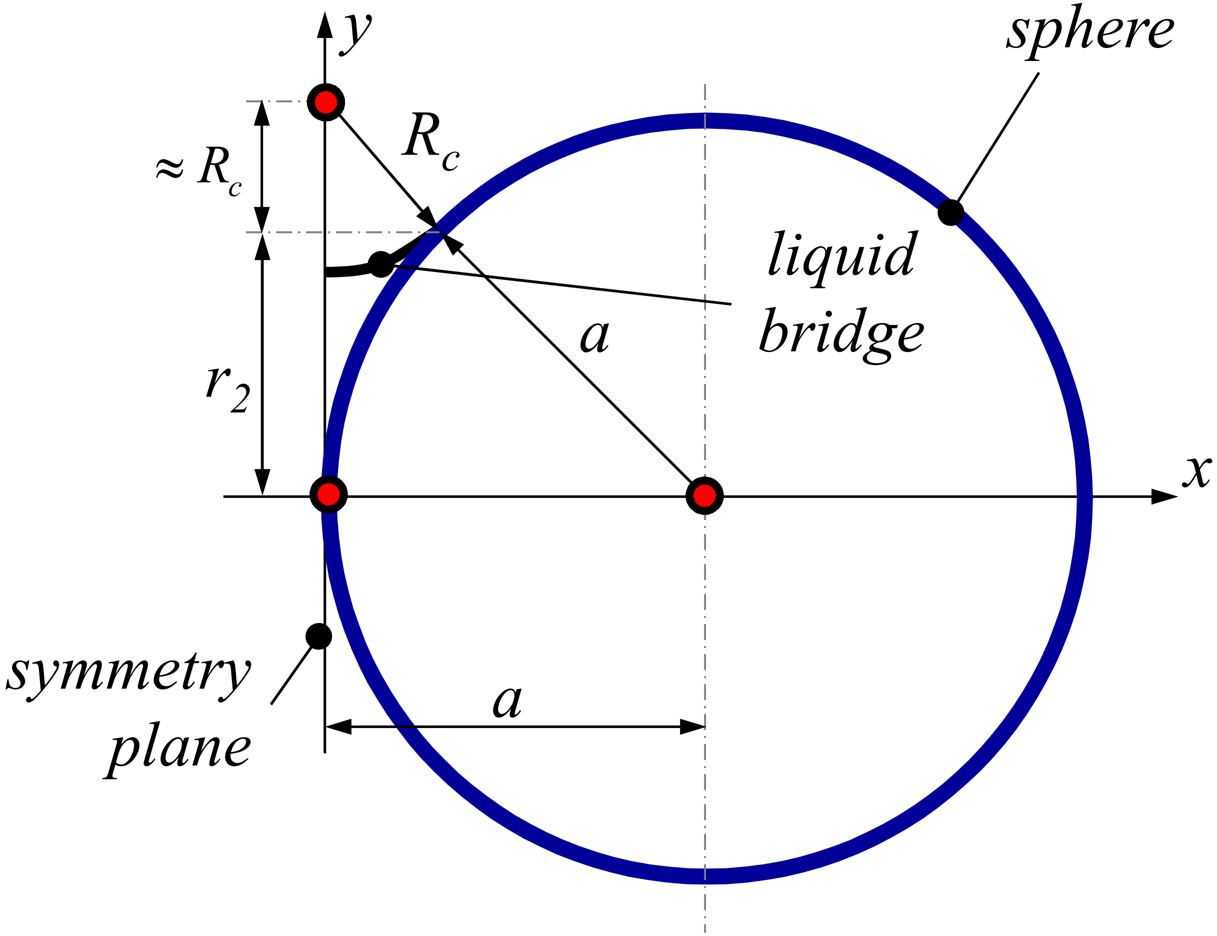
  - **In fact, a somewhat more involved analysis of a (simplified) bridge shape, and for zero separation (see next slide) yields a widely accepted estimate of the liquid bridge force F_{liq} for fully wetted and contacting spheres.**

$$R_c \approx r_2^2 / (2a) \qquad \Delta p \approx \gamma / R_c$$

$$\boxed{F_{liq} \approx \Delta p \, \pi r_2^2 \approx 2\pi \gamma a}$$

Derivation of the Young-Laplace Equation

- **Geometric Approximations used for the Force Estimate**



$$a^2 + (r_2 + R_c)^2 = (a + R_c)^2$$

$$r_2 + R_c \approx r_2$$

$$R_c^{\,2} << a^2$$

$$a^2 + r_2^{\,2} = a^2 + 2\,a\,R_c$$

$$\boxed{r_2^{\,2} \approx 2\,a\,R_c}$$